# Curvature tensors in a 4D Riemann–Cartan space: Irreducible decompositions and superenergy

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Geometric Foundations of Gravity

Geometric Foundations of Gauge Theory

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$$\rightarrow \quad dH = j, \quad H := \partial \mathscr{L} / \partial F = \star F$$

redundancy $A \rightarrow A' = A + d\chi$

conserved external current j

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Complex spinor field:

$$\mathscr{L} = \tfrac{i}{2} \left[ \overline{\Psi} (\star\gamma) \wedge d\Psi - \text{h.c.} \right] + i\, m \star \overline{\Psi} \Psi$$

$$\rightarrow \quad \left[ (\star\gamma) \wedge d + \star m \right] \Psi = 0$$

invariance $\Psi \rightarrow e^{i\alpha} \Psi$, $\overline{\Psi} \rightarrow e^{-i\alpha} \overline{\Psi}$

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conserved $U(1)$ current $j = \overline{\Psi} (\star\gamma) \Psi$

Complete, gauge-theoretical description:

$$\mathscr{L} = \tfrac{1}{2} F \wedge \star F + \tfrac{i}{2} \left[ \overline{\Psi} (\star\gamma) \wedge (d + ieA) \Psi - (d - ieA) \overline{\Psi} \wedge (\star\gamma) \Psi \right] + im \star \overline{\Psi} \Psi$$

- local $U(1)$ invariance $\{ \Psi \to e^{i\alpha(x)} \Psi, \ \overline{\Psi} \to e^{-i\alpha(x)} \overline{\Psi}, \ A \to A + d\alpha(x) \}$

# The ingredients of gauge theory: the example of electrodynamics

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conserved external current j

**theory of force carriers given external current**

Complex spinor field:

$\mathscr{L} = \frac{i}{2}\left[\overline{\Psi}(\star\gamma) \wedge d\Psi - h.c.\right] + i\, m \star \overline{\Psi}\Psi$

$\rightarrow \left[(\star\gamma) \wedge d + m\right]\Psi = 0$

invariance $\Psi \rightarrow e^{i\alpha}\Psi, \ \overline{\Psi} \rightarrow e^{-i\alpha}\overline{\Psi}$

conserved U(1) current $j = \overline{\Psi}(\star\gamma)\Psi$

**microscopic description of matter; Noether currents**

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- local U(1) invariance $\{\Psi \rightarrow e^{i\alpha(x)}\Psi, \ \overline{\Psi} \rightarrow e^{-i\alpha(x)}\overline{\Psi}, \ A \rightarrow A + d\alpha(x)\}$

**gauge theory = complete description of matter and how it interacts via gauge bosons**

# Curvature tensors

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$$\mathrm{R} = \frac{1}{2}\mathrm{R_{ij}}{}^{\mu}{}_{\nu}\,\mathrm{dx^i} \wedge \mathrm{dx^j} \otimes \omega_{\mu}{}^{\nu}$$

# Curvature tensors

$$R = \frac{1}{2} R_{ij}{}^{\mu}{}_{\nu} dx^i \wedge dx^j \otimes \omega_{\mu}{}^{\nu}$$

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# Curvature tensors

$$R = \frac{1}{2} R_{ij}{}^{\mu}{}_{\nu} dx^i \wedge dx^j \otimes \omega_{\mu}{}^{\nu}$$ Riemann curvature tensor

$$T = \frac{1}{2} T_{ij}{}^{\mu} dx^i \wedge dx^j \otimes e_{\mu}$$ Cartan's torsion tensor

# Curvature tensors

$$R = \frac{1}{2} R_{ij}{}^{\mu}{}_{\nu} dx^i \wedge dx^j \otimes \omega_\mu{}^\nu \qquad \text{rotational curvature}$$

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A **Riemann–Cartan geometry $U_4$** is a four-dimensional manifold, whose torsion tensor and curvature tensor satisfy

$$[\nabla_i, \nabla_j] f = T_{ij}{}^{a} \nabla_a f, \qquad [\nabla_i, \nabla_j] V^k = R_{ij}{}^{k}{}_{a} V^a - T_{ij}{}^{a} \nabla_a V^k.$$

The first Bianchi identity $DT^{\mu} = R^{\mu}{}_{\alpha} \wedge \vartheta^{\alpha}$ links dynamical properties of torsion to algebraic properties of curvature.

$\rightarrow$ In the presence of non-vanishing torsion, the curvature tensor has different algebraic properties. Analyze this.

# Young decomposition of a general rank-p tensor

Based on Schur–Weyl duality that links representations of $S_n$ and $GL(4, R)$, see literature.

$$T_{\mu_1\ldots\mu_p} = \bigoplus_{J=1}^{N} {}^{[J]}T_{\mu_1\ldots\mu_p} = \bigoplus_{J=1}^{N} {}^{[J]}\mathbb{P}^{\alpha_1\ldots\alpha_p}_{\mu_1\ldots\mu_p} T_{\alpha_1\ldots\alpha_p}, \quad {}^{[J]}\mathbb{P}^{\alpha_1\ldots\alpha_p}_{\mu_1\ldots\mu_p} := \frac{f^J}{p!}\sum_{k=1}^{f^J} \mathbb{P}^{\alpha_1\ldots\alpha_p}_{\mu_1\ldots\mu_p}\left(Y^J_k\right),$$

$$f^J := \prod_{x\in Y^J} \frac{p!}{\text{hook}(x)}, \quad \text{hook}(x) := \Big(\text{“boxes to the right”} - \text{“boxes below”}\Big)(x) + 1.$$

Here, $Y^J$ is the J-th allowed Young diagram, and $Y^J_k$ is the k-th Young tableaux of the Young diagram $Y^J$. Lastly, $\mathbb{P}^{\alpha_1\ldots\alpha_p}_{\mu_1\ldots\mu_p}\left(Y^J_k\right)$ denotes the Young symmetrizer associated with a certain Young tableaux.

This decomposition is block diagonal in the sense that $T_{\mu_1\ldots\mu_p}T^{\mu_1\ldots\mu_p} = \bigoplus_{J=1}^{N} {}^{[J]}T_{\mu_1\ldots\mu_p}{}^{[J]}T^{\mu_1\ldots\mu_p}$.

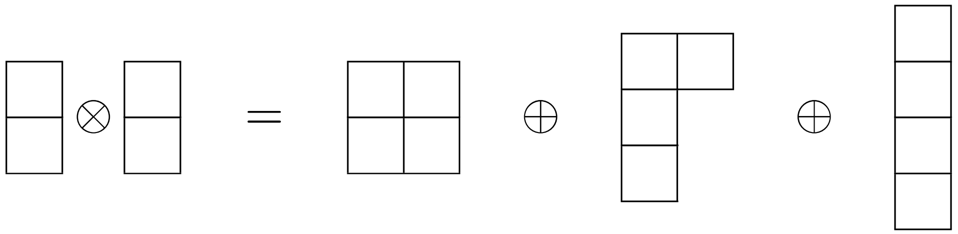
$\rightarrow$ Let us apply this to the Riemann tensor of a $U_4$ geometry with curvature and torsion!

# Young decomposition of the Riemann curvature tensor (1/2)

Symmetries of the Riemann tensor: #

- double 2-form: $R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho}$ (algebraic curvature tensor) 36
- Bianchi identity $R^{\mu}{}_{[\nu\rho\sigma]} = 0$ (if torsion vanishes) 16
- implications: $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}\,,\ R_{[\mu\nu\rho\sigma]} = 0$ (if torsion vanishes) $15 + 1$

Young decomposition of the Riemann tensor ( $6 \times 6 = 36 = 20 \oplus 15 \oplus 1$ ):



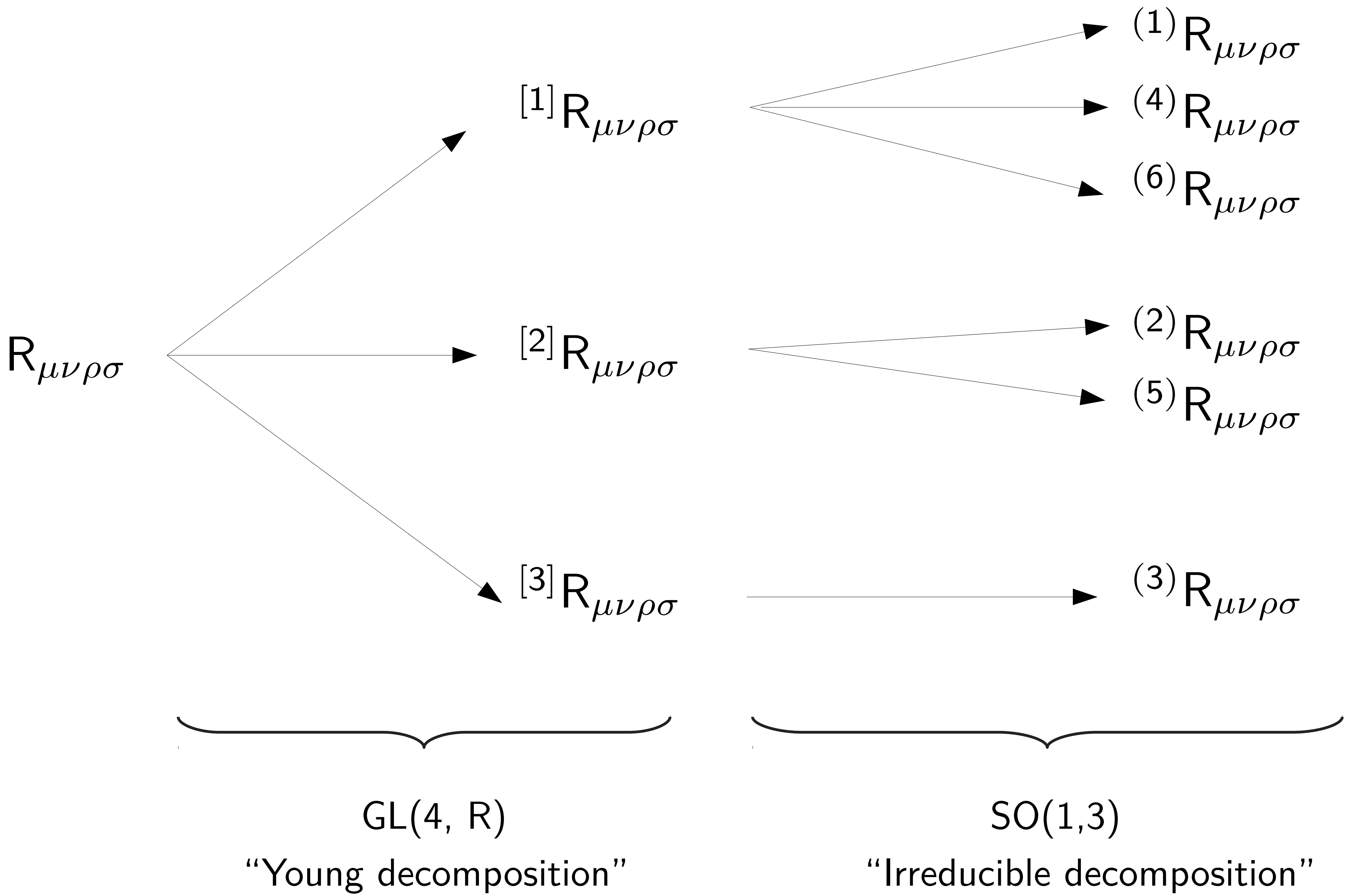
$$R_{\mu\nu\rho\sigma} = \underbrace{\frac{1}{2}\left(R_{\mu\nu\rho\sigma} + R_{\rho\sigma\mu\nu}\right)} \oplus \underbrace{\left[\frac{1}{2}\left(R_{\mu\nu\rho\sigma} - R_{\rho\sigma\mu\nu}\right) - R_{[\mu\nu\rho\sigma]}\right]} \oplus R_{[\mu\nu\rho\sigma]}$$

Weyl tensor
symmetric tracefree Ricci tensor
Ricci scalar

“paircom” tensor
antisymmetric Ricci tensor

pseudoscalar

# Young decomposition of the Riemann curvature tensor (2/2)

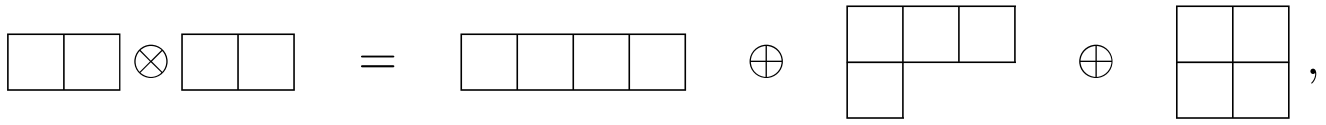


# An algebraic superenergy tensor in Poincaré gauge theory of gravity (1/3)

The **Bel tensor** can be defined in terms of the duals of the Riemann tensor:

$$B_{\mu\nu\rho\sigma} := \frac{1}{2}\Big(R_{\mu\alpha\beta\rho}R_{\nu}{}^{\alpha\beta}{}_{\sigma} + (*R*)_{\mu\alpha\beta\rho}(*R*)_{\nu}{}^{\alpha\beta}{}_{\sigma} + (*R)_{\mu\alpha\beta\rho}(*R)_{\nu}{}^{\alpha\beta}{}_{\sigma} + (R*)_{\mu\alpha\beta\rho}(R*)_{\nu}{}^{\alpha\beta}{}_{\sigma}\Big)$$

The Young decomposition is



$$B_{\mu\nu\rho\sigma} = B_{(\mu\nu\rho\sigma)} \oplus \frac{1}{2}\left(B_{\mu\nu\rho\sigma} - B_{\rho\sigma\mu\nu}\right) \oplus \frac{1}{6}\Big[2\left(B_{\mu\nu\rho\sigma} + B_{\rho\sigma\mu\nu}\right) - \left(B_{\mu\rho\nu\sigma} + B_{\nu\sigma\mu\rho}\right) - \left(B_{\mu\sigma\nu\rho} + B_{\nu\rho\mu\sigma}\right)\Big].$$

In General Relativity, the **Bel–Robinson tensor** is constructed analogously from the Weyl tensor. It is also related to superenergy: a positive definite quantity for a timelike observer. How to generalize to Poincaré gauge theory?

→ Introduce Bel trace tensor $B_{\mu\nu} := B^{\alpha}{}_{\mu\alpha\nu}$ and subtract traces to define an **algebraic Bel–Robinson tensor**.

# An algebraic superenergy tensor in Poincaré gauge theory of gravity (2/3)

Explicit form of the decomposition of the Bel tensor:

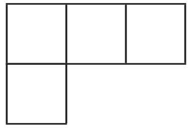
$$^{(1b)}\mathrm{B}_{\mu\nu\rho\sigma} := \frac{1}{12}\left(g_{\mu\nu}\not{\mathrm{B}}_{\rho\sigma} + g_{\rho\sigma}\not{\mathrm{B}}_{\mu\nu} + g_{\mu\rho}\not{\mathrm{B}}_{\nu\sigma} + g_{\nu\sigma}\not{\mathrm{B}}_{\mu\rho} + g_{\mu\sigma}\not{\mathrm{B}}_{\nu\rho} + g_{\nu\rho}\not{\mathrm{B}}_{\mu\sigma}\right),$$

$$^{(1c)}\mathrm{B}_{\mu\nu\rho\sigma} := \frac{1}{36}\mathrm{B}\left(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\right),$$

$$^{(1a)}\mathrm{B}_{\mu\nu\rho\sigma} := {}^{[1]}\mathrm{B}_{\mu\nu\rho\sigma} - {}^{(1b)}\mathrm{B}_{\mu\nu\rho\sigma} - {}^{(1c)}\mathrm{B}_{\mu\nu\rho\sigma},$$

$$^{(2b)}\mathrm{B}_{\mu\nu\rho\sigma} := \frac{1}{6}\left(g_{\mu\rho}\mathrm{B}_{[\nu\sigma]} + g_{\nu\rho}\mathrm{B}_{[\mu\sigma]} + g_{\mu\sigma}\mathrm{B}_{[\nu\rho]} + g_{\nu\sigma}\mathrm{B}_{[\mu\rho]}\right),$$

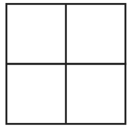
$$^{(2a)}\mathrm{B}_{\mu\nu\rho\sigma} := {}^{[2]}\mathrm{B}_{\mu\nu\rho\sigma} - {}^{(2b)}\mathrm{B}_{\mu\nu\rho\sigma},$$



$$^{(3b)}\mathrm{B}_{\mu\nu\rho\sigma} := \frac{1}{6}\left(g_{\mu\rho}\not{\mathrm{B}}_{\nu\sigma} + g_{\nu\sigma}\not{\mathrm{B}}_{\mu\rho} + g_{\mu\sigma}\not{\mathrm{B}}_{\nu\rho} + g_{\nu\rho}\not{\mathrm{B}}_{\mu\sigma} - 2g_{\mu\nu}\not{\mathrm{B}}_{\rho\sigma} - 2g_{\rho\sigma}\not{\mathrm{B}}_{\mu\nu}\right),$$

$$^{(3c)}\mathrm{B}_{\mu\nu\rho\sigma} := \frac{1}{36}\mathrm{B}\left(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho} - 2g_{\mu\nu}g_{\rho\sigma}\right),$$

$$^{(3a)}\mathrm{B}_{\mu\nu\rho\sigma} := {}^{[3]}\mathrm{B}_{\mu\nu\rho\sigma} - {}^{(3b)}\mathrm{B}_{\mu\nu\rho\sigma} - {}^{(3c)}\mathrm{B}_{\mu\nu\rho\sigma}.$$



# An algebraic superenergy tensor in Poincaré gauge theory of gravity (3/3)

The following is our **final result**:

$$
\begin{aligned}
{}^{(1a)}\mathrm{B}_{\mu\nu\rho\sigma} &:= {}^{[1]}\mathrm{B}_{\mu\nu\rho\sigma} - {}^{(1b)}\mathrm{B}_{\mu\nu\rho\sigma} - {}^{(1c)}\mathrm{B}_{\mu\nu\rho\sigma},\\
{}^{(1b)}\mathrm{B}_{\mu\nu\rho\sigma} &:= \frac{1}{12}\left(g_{\mu\nu}\not{\mathrm{B}}_{\rho\sigma} + g_{\rho\sigma}\not{\mathrm{B}}_{\mu\nu} + g_{\mu\rho}\not{\mathrm{B}}_{\nu\sigma} + g_{\nu\sigma}\not{\mathrm{B}}_{\mu\rho} + g_{\mu\sigma}\not{\mathrm{B}}_{\nu\rho} + g_{\nu\rho}\not{\mathrm{B}}_{\mu\sigma}\right),\\
{}^{(1c)}\mathrm{B}_{\mu\nu\rho\sigma} &:= \frac{1}{36}\mathrm{B}\left(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{B}_{\mu\nu} &:= \mathrm{B}^{\alpha}{}_{\mu\alpha\nu} =: \not{\mathrm{B}}_{\mu\nu} \oplus \mathrm{B}_{[\mu\nu]} \oplus \frac{1}{4}\mathrm{B}g_{\mu\nu},\\
\not{\mathrm{B}}_{\mu\nu} &= {}^{(2)}\mathrm{R}_{\mu\alpha\beta\gamma}{}^{(2)}\mathrm{R}^{\alpha\beta\gamma}{}_{\nu} - g^{\alpha\beta}\left(2\mathrm{Ric}_{[\mu\alpha]}\mathrm{Ric}_{[\nu\beta]} + \not{\mathrm{Ric}}_{\mu\alpha}\not{\mathrm{Ric}}_{\nu\beta}\right)\\
&\quad + \frac{1}{4}g_{\mu\nu}\left(2\mathrm{Ric}_{[\alpha\beta]}\mathrm{Ric}^{[\alpha\beta]} + \not{\mathrm{Ric}}_{\alpha\beta}\not{\mathrm{Ric}}^{\alpha\beta}\right),\\
\mathrm{B}_{[\mu\nu]} &= \frac{1}{2}\left(\mathrm{R}\mathrm{Ric}_{[\mu\nu]} + \frac{1}{2}\chi\eta_{\mu\nu\alpha\beta}\mathrm{Ric}^{[\alpha\beta]}\right),\\
{}^{(3)}\mathrm{B}_{\mu\nu} &= \frac{1}{4}\left(-\frac{1}{2}{}^{(2)}\mathrm{R}_{\alpha\beta\gamma\delta}{}^{(2)}\mathrm{R}^{\alpha\beta\gamma\delta} + \not{\mathrm{Ric}}_{\alpha\beta}\not{\mathrm{Ric}}^{\alpha\beta} + \frac{1}{4}\mathrm{R}^2 + \frac{1}{4}\chi^2\right)g_{\mu\nu}.
\end{aligned}
$$

## Some remarks on the Bel trace tensor

The Bel trace tensor lists how different curvature ingredients contribute to traces:

- In General Relativity, $\mathrm{Ric}_{\mu\nu} = 0$ implies $\mathrm{B}_{\mu\nu} = 0$ .
- In other theories (different Lagrangian, different geometry with torsion, ...), the vacuum field equations may impose other constraints on the curvature.
- Only the Weyl tensor does not appear in the Bel trace tensor. This is because it is traceless, ${}^{(1)}\mathrm{R}^{\alpha}{}_{\mu\alpha\beta} = 0$ , and it also satisfies ${}^{(1)}\mathrm{R}^{\mu}{}_{[\nu\rho\sigma]} = 0$ .

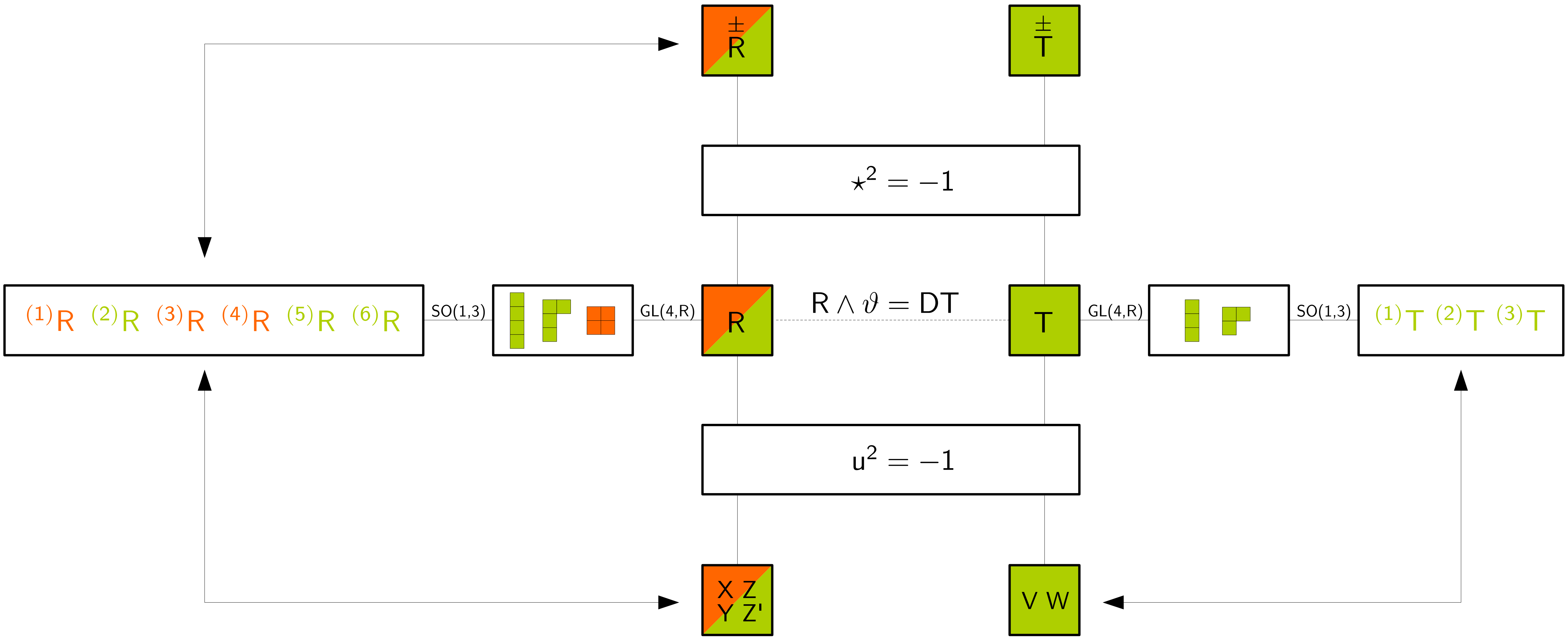
## Conclusions

The Bel trace tensor allows us to define a tensor that has the same algebraic properties as the Bel–Robinson tensor. Further work needs to be done:

- Would a spinorial treatment give rise to a deeper algebraic understanding?
- What about differential properties of the algebraic Bel–Robinson tensor?

**Thank you for your attention.**

# Outlook: further decompositions in four dimensions




$V_4$ geometry with vanishing torsion. $U_4$ geometry with non-vanishing torsion.