Poincaré gauge theory and its deformed Lie algebra – mass-spin classification of elementary particles

Jens Boos
jboos@perimeterinstitute.ca
Perimeter Institute for Theoretical Physics

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Introduction and outline

Question: What do quantum field theory and gravity have in common?

Quantum field theory:

- states in a Hilbert space,
- field operators on Minkowski space

Gravity:

- dynamic spacetime
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Outline:

0. The Poincaré group
1. Quantum field theory
2. Poincaré gauge theory of gravity
3. Putting it all together

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Isometry group of Minkowski space. Noether's theorem predicts conserved energy-momentum, angular momentum, and orbital angular momentum.

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Lie algebra:
$$[M_{\mu\nu}, M_{\alpha\beta}] = g_{\alpha[\mu}M_{\nu]\beta} - g_{\beta[\mu}M_{\nu]\alpha},$$
$$[M_{\mu\nu}, P_{\alpha}] = g_{\alpha[\mu}P_{\nu]},$$
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Casimir operators: $C_1 := P_\alpha P^\alpha, \quad C_2 := W_\alpha W^\alpha$

$\rightarrow$ values are independent of representation (scalars)

Pauli–Lubanski pseudovector (in 4D only): $W_\mu := -\frac{1}{2}\epsilon_{\mu\alpha\beta\gamma}M^{\alpha\beta}P^\gamma$

1. QFT: Wigner's little group

Why is the Poincaré group important in particle physics?

$\rightarrow$ Wigner (1939): it allows us to **invariantly** classify one-particle states $|\mathrm{p}, \sigma\rangle$ (momentum eigenvalues given by $\hat{\mathrm{P}}_\mu |\mathrm{p}, \sigma\rangle = \mathrm{p}_\mu |\mathrm{p}, \sigma\rangle$)

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Example:
- $p^2 = -m^2$ (massive particle): $W = SO(3) \rightarrow$ spin as quantum number
- $p^2 = 0$ (massless particle): $W = SE(2) \rightarrow$ helicity as quantum number (or continuous spin particles)

1. QFT: Relativistic wave equations

Suitable real-space representation of the Poincaré algebra: $\mathsf{M}_{\mathsf{ij}} \sim \mathsf{x}_{[\mathsf{i}}\partial_{\mathsf{j}]}$, $\mathsf{P}_{\mathsf{a}} \sim \partial_{\mathsf{a}}$

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How can that all be related to gravity?

- Poincaré group is the isometry group of Minkowski space
  $\rightarrow$ cannot simply be extended to a curved background

- let us consider **Poincaré gauge theory** as a viable theory of gravity

2. Poincaré gauge theory of gravity (1/2)

"Newton successfully wrote apple = moon, but you cannot write apple = neutron."

– J. L. Synge

Consider a matter field (e.g. Dirac spinor) on Minkowski background:

$$\mathcal{L} = \bar{\Psi}\left(i\gamma^j\partial_j - m\right)\Psi$$

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Make this theory invariant under local Poincaré transformations:

$$\mathcal{L} = \bar{\Psi}\left[\mathrm{i}\, e^{j}{}_{\alpha}(x)\gamma^{\alpha}\left(\partial_{j} + \frac{\mathrm{i}}{4}\Gamma_{j}(x)\right) - m\right]\Psi$$

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Compared to Minkowski space, we are forced to introduce gauge potentials:

$$\begin{aligned} \delta^i_\mu &\mapsto e^i{}_\mu(x), && = 4 \text{ translational gauge potentials} \\ \partial_i &\mapsto D_i := \partial_i - \Gamma_i{}^{\alpha\beta}(x) f_{\alpha\beta} && = 6 \text{ rotational gauge potentials} \end{aligned}$$

2. Poincaré gauge theory of gravity (2/2)

Field strengths related to the potentials $e_j{}^{\mu}$ and $\Gamma_j{}^{\mu\nu}$:

$$F_{ij}{}^{\mu} := 2\left(\partial_{[i}e_{j]}{}^{\mu} + \Gamma_{[i}{}^{\mu\alpha}e_{j]}{}^{\beta}g_{\alpha\beta}\right) \quad = \text{translational curvature}$$

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This class of theories indeed describes gravity (e.g. Einstein–Cartan theory) and includes General Relativity in the limit $F_{ij}{}^{\mu} \overset{!}{=} 0$ with the Lagrangian $F := e^{i}{}_{\mu}e^{j}{}_{\nu}F_{ij}{}^{\mu\nu}$.

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Translation into differential geometry:

- Latin indices ↔ coordinate indices, Greek indices ↔ orthonormal frame indices
- gauge potentials $e_j{}^\mu$ and $\Gamma_j{}^{\mu\nu}$ correspond to tetrad and connection
- gauge curvatures $F_{ij}{}^{\mu}$ and $F_{ij}{}^{\mu\nu}$ correspond to torsion and curvature

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Note that the Poincaré transformations act on space itself (not some internal space):

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[M_{\mu\nu}, M_{\alpha\beta}] &= g_{\alpha[\mu}M_{\nu]\beta} - g_{\beta[\mu}M_{\nu]\alpha}, \\
[M_{\mu\nu}, D_{\alpha}] &= g_{\alpha[\mu}D_{\nu]}, \\
[D_{\mu}, D_{\alpha}] &= F_{\mu\alpha}{}^{\rho\sigma}M_{\rho\sigma} - F_{\mu\alpha}{}^{\sigma}D_{\sigma} \neq 0.
\end{aligned}$$

This deformed Lie algebra is unique to external gauge theories.

3. Putting it all together?

Deformed Lie algebra:

$$[M_{\mu\nu}, M_{\alpha\beta}] = g_{\alpha[\mu}M_{\nu]\beta} - g_{\beta[\mu}M_{\nu]\alpha},$$
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$$[D_\mu, D_\alpha] = F_{\mu\alpha}{}^{\rho\sigma}M_{\rho\sigma} - F_{\mu\alpha}{}^{\sigma}D_\sigma \neq 0.$$

Questions:

- What is the resulting group?
- How do the Casimir operators look like?
- What does that tell us about particles on a curved background?
- ...

Thank you for your attention.

References

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Appendix: conserved Noether currents for the Poincaré group

Conserved Noether currents:

energy-momentum: $\partial_i \mathfrak{T}_k{}^i = 0$ $\qquad \mathfrak{T}^i{}_k := \mathcal{L}\delta^i_k - \frac{\partial \mathcal{L}}{\partial\left(\partial_i \Phi^A\right)}\left(\partial_k \Phi^A\right)$

total angular momentum: $\partial_i \left(\mathfrak{S}_{kl}{}^i + x_{[k}\mathfrak{T}_{l]}{}^i\right) = 0$ $\qquad \mathfrak{S}_{kl}{}^i := \frac{\partial \mathcal{L}}{\partial\left(\partial_i \Phi^A\right)} f_{[kl]}{}^A{}_B \Phi^B$

Note: the semidirect product structure is everywhere.