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8. Differential forms

A differential form of rank p ("p-form") is a completely antisymmetric tensor of rank $\binom{0}{p}$.

Example: $F_{\mu\nu} = -F_{\nu\mu}$ 2-form

$\varepsilon_{\mu\nu\rho\sigma} = -\varepsilon_{\nu\mu\rho\sigma} = -\varepsilon_{\rho\mu\nu\sigma} = \ldots$ 4-form

A_μ 1-form

ϕ 0-form

Wedge product:

$$\underline{F} = F_{\mu\nu}\, dx^\mu \otimes dx^\nu = \frac{1}{2}(F_{\mu\nu} - F_{\nu\mu})\, dx^\mu \otimes dx^\nu$$

$$= \frac{1}{2} F_{\mu\nu}\, dx^\mu \otimes dx^\nu - \frac{1}{2} F_{\nu\mu}\, dx^\mu \otimes dx^\nu = \frac{1}{2} F_{\mu\nu}\, dx^\mu \otimes dx^\nu - \frac{1}{2} F_{\mu\nu}\, dx^\nu \otimes dx^\mu$$

$$= F_{\mu\nu}\, \frac{1}{2}(dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu) \equiv F_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

Differential forms of rank p span a vector space Λ^p.

Wedge product is a map from $\Lambda^p \times \Lambda^q \to \Lambda^{p+q}$. ("Wedge product = totally antisymm. tensor product")

General rule: $\underline{\omega}$ is a p-form, $\underline{\lambda}$ is a q-form, then $\underline{\omega} \wedge \underline{\lambda} = (-1)^{pq}\, \underline{\lambda} \wedge \underline{\omega}$.

Examples: $\underline{F} = dx \wedge dy$, $\underline{H} = dt \wedge dz + dx \wedge dy$, $\underline{A} = dt$, $\underline{B} = dx$

$$\underline{A} \wedge \underline{F} = dt \wedge dx \wedge dy = -dx \wedge dt \wedge dy = +dx \wedge dy \wedge dt = \underline{F} \wedge \underline{A}$$

$$\underline{A} \wedge \underline{B} = dt \wedge dx = -dx \wedge dt = -\underline{B} \wedge \underline{A}$$

$$\underline{B} \wedge \underline{B} = dx \wedge dx = 0$$

$$\underline{H} \wedge \underline{H} = (dt \wedge dz + dx \wedge dy) \wedge (dt \wedge dz + dx \wedge dy) = 2\, dt \wedge dz \wedge dx \wedge dy$$

Why are differential forms so useful?

→ We can differentiate them in a meaningful way <u>without a connection</u>!

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Exterior derivative!

$$\partial_\rho \omega_{\mu\nu} \longrightarrow \frac{\partial y^{\rho'}}{\partial x^\rho} \partial_{\rho'} \left(\frac{\partial y^{\mu'}}{\partial x^\mu} \frac{\partial y^{\nu'}}{\partial x^\nu} \omega_{\mu'\nu'} \right)$$

$$= \frac{\partial y^{\rho'}}{\partial x^\rho} \frac{\partial y^{\mu'}}{\partial x^\mu} \frac{\partial y^{\nu'}}{\partial y^\nu} \partial_{\rho'} \omega_{\mu'\nu'} + \left[\underbrace{\frac{\partial^2 y^{\mu'}}{\partial x^\rho \partial x^\mu}}_{\substack{\text{symmetric} \\ \text{in } \rho \leftrightarrow \mu}} \frac{\partial y^{\nu'}}{\partial x^\nu} + \frac{\partial y^{\mu'}}{\partial x^\mu} \underbrace{\frac{\partial^2 y^{\nu'}}{\partial x^\rho \partial x^\nu}}_{\substack{\text{symmetric} \\ \text{in } \rho \leftrightarrow \nu}} \right] \omega_{\mu'\nu'}$$

The inhomogeneous non-tensorial part <u>cancels out</u> if we antisymmetrize between $\rho \leftrightarrow \mu$ and $\rho \leftrightarrow \nu$. Together with existing antisymmetry $\omega_{\mu\nu} = -\omega_{\nu\mu}$ this amounts to <u>total</u> antisymmetrization.

$$\partial_{[\rho} \omega_{\mu\nu]} \longrightarrow \frac{\partial y^{\rho'}}{\partial x^\rho} \frac{\partial y^{\mu'}}{\partial x^\mu} \frac{\partial y^{\nu'}}{\partial x^\nu} \partial_{[\rho'} \omega_{\mu'\nu']} \qquad \text{tensorial transformation!}$$

Notation: $[\mu\nu\rho] = \frac{1}{3!} \left(\mu\nu\rho - \mu\rho\nu + \rho\mu\nu - \rho\nu\mu + \nu\rho\mu - \nu\mu\rho \right)$

For any p-form, we define the <u>exterior derivative</u>:

$$d\,\underline{\omega} \equiv d \left(\omega_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \right)$$

$$= \left(\partial_\rho \omega_{\mu_1 \cdots \mu_p} \right) dx^\rho \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \qquad \hat{=} \ (p+1)\text{-form.}$$

$$(d\underline{\omega})_{\mu_1 \cdots \mu_{p+1}} = \partial_{[\mu_1} \omega_{\mu_2 \cdots \mu_{p+1}]}$$

<u>Example</u>: $\underline{\omega} = 27\,xy\,dy \wedge dz$

$$d\underline{\omega} = 27y\,dx \wedge dy \wedge dz + 27x \underbrace{dy \wedge dy \wedge dz}_{=0} + 0$$

$$= 27y\,dx \wedge dy \wedge dz$$

<u>Properties</u>: $\underline{\omega}$ p-form, $\underline{\lambda}$ q-form

$d(d\underline{\omega}) = 0$, $\quad d(\underline{\omega} \wedge \underline{\lambda}) = (d\underline{\omega}) \wedge \underline{\lambda} + (-1)^p \underline{\omega} \wedge (d\underline{\lambda})$ $\qquad$ Leibniz