7. Riemannian curvature

Levi-Civita connection $\tilde{\Gamma}^\mu{}_{\nu\rho} = \frac{1}{2} g^{\mu\alpha}(\partial_\nu g_{\alpha\rho} + \partial_\rho g_{\alpha\nu} - \partial_\alpha g_{\nu\rho})$

Sometimes also called "Christoffel symbols."

Denote LC-covariant derivative as $\tilde{\nabla}_\rho T^\mu{}_\nu = \partial_\rho T^\mu{}_\nu + \tilde{\Gamma}^\mu{}_{\rho\alpha} T^\alpha{}_\nu - \tilde{\Gamma}^\alpha{}_{\rho\nu} T^\mu{}_\alpha$

Riemann curvature tensor: $[\tilde{\nabla}_\alpha, \tilde{\nabla}_\beta] V^\mu = \tilde{R}_{\alpha\beta}{}^\mu{}_\nu V^\nu - 0$ ← no torsion in the Levi-Civita connection!

General curvature tensor: $[\nabla_\alpha, \nabla_\beta] V^\mu = R_{\alpha\beta}{}^\mu{}_\nu V^\nu - T_{\alpha\beta}{}^\lambda \nabla_\lambda V^\mu$

Some properties of the curvature tensor:

$$\tilde{R}_{\alpha\beta}{}^\mu{}_\nu = \partial_\alpha \tilde{\Gamma}^\mu{}_{\beta\nu} - \partial_\beta \tilde{\Gamma}^\mu{}_{\alpha\nu} + \tilde{\Gamma}^\mu{}_{\alpha\lambda}\tilde{\Gamma}^\lambda{}_{\beta\nu} - \tilde{\Gamma}^\mu{}_{\beta\lambda}\tilde{\Gamma}^\lambda{}_{\alpha\nu}$$

$$\begin{aligned}
\tilde{R}_{\alpha\beta\mu\nu} &= g_{\mu\rho}\tilde{R}_{\alpha\beta}{}^\rho{}_\nu \\
&= \partial_\alpha(g_{\mu\rho}\tilde{\Gamma}^\rho{}_{\beta\nu}) - \tilde{\Gamma}^\rho{}_{\beta\nu}(\partial_\alpha g_{\mu\rho}) - \partial_\beta(g_{\mu\rho}\tilde{\Gamma}^\rho{}_{\alpha\nu}) + \tilde{\Gamma}^\rho{}_{\alpha\nu}(\partial_\beta g_{\mu\rho}) \\
&\quad + \tilde{\Gamma}_{\mu\alpha\lambda}\tilde{\Gamma}^\lambda{}_{\beta\nu} - \tilde{\Gamma}_{\mu\beta\lambda}\tilde{\Gamma}^\lambda{}_{\alpha\nu} \\
&= \partial_\alpha \tilde{\Gamma}_{\mu\beta\nu} - \partial_\beta \tilde{\Gamma}_{\mu\alpha\nu} - \tilde{\Gamma}^\rho{}_{\beta\nu}(\cancel{\tilde{\Gamma}_{\mu\alpha\rho}}^{①} + \tilde{\Gamma}_{\rho\alpha\mu}) + \tilde{\Gamma}^\rho{}_{\alpha\nu}(\cancel{\tilde{\Gamma}_{\mu\beta\rho}}^{②} + \tilde{\Gamma}_{\rho\beta\mu}) \\
&\quad + \cancel{\tilde{\Gamma}_{\mu\alpha\lambda}}^{①}\tilde{\Gamma}^\lambda{}_{\beta\nu} - \cancel{\tilde{\Gamma}_{\mu\beta\lambda}}^{②}\tilde{\Gamma}^\lambda{}_{\alpha\nu} \\
&= \partial_\alpha \tilde{\Gamma}_{\mu\beta\nu} - \partial_\beta \tilde{\Gamma}_{\mu\alpha\nu} + \tilde{\Gamma}^\lambda{}_{\alpha\nu}\tilde{\Gamma}_{\lambda\beta\mu} - \tilde{\Gamma}^\lambda{}_{\beta\nu}\tilde{\Gamma}_{\lambda\alpha\mu}
\end{aligned}$$

Can check: $\tilde{R}_{\alpha\beta\mu\nu} = -\tilde{R}_{\beta\alpha\mu\nu}$, $\tilde{R}_{\alpha\beta\mu\nu} = -\tilde{R}_{\alpha\beta\nu\mu}$ } two pairs of "antisymmetric" indices $[\alpha\beta]$ and $[\mu\nu]$

$\tilde{R}_{\alpha[\beta\mu\nu]} = 0$ → Bianchi identity (found by Ricci)

$\Leftrightarrow$ $\tilde{R}_{\alpha\beta\mu\nu} = \tilde{R}_{\mu\nu\alpha\beta} \wedge R_{[\mu\nu\alpha\beta]} = 0$

→ # of independent components of $\tilde{R}_{\alpha\beta\mu\nu}$ is $\frac{n^2(n^2-1)}{12}$

General $R_{\alpha\beta\mu\nu}$ with torsion: only have antisymmetry $\alpha \leftrightarrow \beta$ and $\mu \leftrightarrow \nu$

$$\rightarrow \# = \frac{n^2(n-1)^2}{4}$$

Ricci tensor:

$$\tilde{R}_{\mu\nu} \equiv \tilde{R}_{\alpha\nu}{}^{\alpha}{}_{\mu} = \delta^{\sigma}_{\rho} \tilde{R}_{\sigma\nu}{}^{\rho}{}_{\mu} = \tilde{R}_{\nu\mu}$$

Ricci scalar:

$$\tilde{R} = g^{\mu\nu} \tilde{R}_{\mu\nu}$$

Let's calculate the scalar curvature of a cylinder to prove it is flat!

$$\underline{g} = dr \otimes dr + r^2 d\varphi \otimes d\varphi + dz \otimes dz \Big|_{r=\text{const.}} = \underset{\nwarrow \text{ constant!}}{r^2} d\varphi \otimes d\varphi + dz \otimes dz$$

$$\tilde{\Gamma}^{\mu}{}_{\nu\rho} = \frac{1}{2} g^{\mu\alpha} \left(\partial_\nu g_{\alpha\rho} + \partial_\rho g_{\alpha\nu} - \partial_\alpha g_{\nu\rho} \right)$$

$= 0$ b/c $g_{\mu\nu}$ = constant since r = constant!

$\Rightarrow \tilde{R}_{\alpha\beta}{}^{\mu}{}_{\nu} = 0 \Rightarrow$ cylinder is flat!

Let's show that the scalar curvature of the 2-sphere is related to its radius!

$$\underline{g} = dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2\theta \, d\varphi \otimes d\varphi \Big|_{r=\text{const.}} = r^2 d\theta \otimes d\theta + r^2 \sin^2\theta \, d\varphi \otimes d\varphi$$

r = constant!

$$\tilde{\Gamma}^{\theta}{}_{\theta\theta} = \frac{1}{2} g^{\theta\theta} \left(\partial_\theta g_{\theta\theta} + \partial_\theta g_{\theta\theta} - \partial_\theta g_{\theta\theta} \right) = 0$$

$$\tilde{\Gamma}^{\theta}{}_{\theta\varphi} = \frac{1}{2} g^{\theta\theta} \left(\partial_\theta g_{\theta\varphi} + \partial_\varphi g_{\theta\theta} - \partial_\theta g_{\theta\varphi} \right) = 0$$

$$\tilde{\Gamma}^{\theta}{}_{\varphi\varphi} = \frac{1}{2} g^{\theta\theta} \left(\partial_\varphi g_{\theta\varphi} + \partial_\varphi g_{\theta\varphi} - \partial_\theta g_{\varphi\varphi} \right) = -\frac{1}{2} \frac{1}{r^2} r^2 2 \sin\theta \cos\theta = -\sin\theta\cos\theta$$

$$\tilde{\Gamma}^{\varphi}{}_{\theta\theta} = \frac{1}{2} g^{\varphi\varphi} \left(\partial_\theta g_{\varphi\theta} + \partial_\theta g_{\varphi\theta} - \partial_\varphi g_{\theta\theta} \right) = 0$$

$$\tilde{\Gamma}^{\varphi}{}_{\theta\varphi} = \frac{1}{2} g^{\varphi\varphi} \left(\partial_\theta g_{\varphi\varphi} + \partial_\varphi g_{\varphi\theta} - \partial_\varphi g_{\theta\varphi} \right) = \frac{1}{2} \frac{1}{r^2 \sin^2\theta} r^2 2 \sin\theta\cos\theta = \frac{\cos\theta}{\sin\theta}$$

$$\tilde{\Gamma}^{\varphi}{}_{\varphi\varphi} = \frac{1}{2} g^{\varphi\varphi} \left(\partial_\varphi g_{\varphi\varphi} + \partial_\varphi g_{\varphi\varphi} - \partial_\varphi g_{\varphi\varphi} \right) = 0$$

$\rightarrow$ only 2 non-zero $\tilde{\Gamma}^{\mu}{}_{\nu\rho}$.

Next: calculate $\tilde{R}_{\alpha\beta}{}^{\mu}{}_{\nu}$

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$$\tilde{R}_{\alpha\beta}{}^{\mu}{}_{\nu} = \partial_\alpha \tilde{\Gamma}^{\mu}{}_{\beta\nu} - \partial_\beta \tilde{\Gamma}^{\mu}{}_{\alpha\nu} + \tilde{\Gamma}^{\mu}{}_{\alpha\lambda}\tilde{\Gamma}^{\lambda}{}_{\beta\nu} - \tilde{\Gamma}^{\mu}{}_{\beta\lambda}\tilde{\Gamma}^{\lambda}{}_{\alpha\nu}$$

We know there is only 1 component that matters b/c $\frac{n^2(n^2-1)}{12} = 1$ for $n=2$.

$$\tilde{R}_{\theta\varphi}{}^{\theta}{}_{\varphi} = \partial_\theta \tilde{\Gamma}^{\theta}{}_{\varphi\varphi} - \partial_\varphi \tilde{\Gamma}^{\theta}{}_{\theta\varphi} + \tilde{\Gamma}^{\theta}{}_{\theta\lambda}\tilde{\Gamma}^{\lambda}{}_{\varphi\varphi} - \tilde{\Gamma}^{\theta}{}_{\varphi\lambda}\tilde{\Gamma}^{\lambda}{}_{\theta\varphi}$$

$$= \partial_\theta \tilde{\Gamma}^{\theta}{}_{\varphi\varphi} - 0 + \underbrace{\tilde{\Gamma}^{\theta}{}_{\theta\theta}}_{=0}\tilde{\Gamma}^{\theta}{}_{\varphi\varphi} + \tilde{\Gamma}^{\theta}{}_{\theta\varphi}\underbrace{\tilde{\Gamma}^{\varphi}{}_{\varphi\varphi}}_{=0} - \underbrace{\tilde{\Gamma}^{\theta}{}_{\varphi\theta}}_{=0}\tilde{\Gamma}^{\theta}{}_{\theta\varphi} - \tilde{\Gamma}^{\theta}{}_{\varphi\varphi}\tilde{\Gamma}^{\varphi}{}_{\theta\varphi}$$

$$= \partial_\theta(-\sin\theta\cos\theta) - (-\sin\theta\cos\theta)\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$= -\cos^2\theta + \sin^2\theta + \cos^2\theta = \sin^2\theta$$

$$\tilde{R}_{\mu\nu} = \tilde{R}_{\alpha\mu}{}^{\alpha}{}_{\nu} = \tilde{R}_{\theta\mu}{}^{\theta}{}_{\nu} + \tilde{R}_{\varphi\mu}{}^{\varphi}{}_{\nu}$$

$$\tilde{R}_{\theta\theta} = \tilde{R}_{\theta\theta}{}^{\theta}{}_{\theta} + \tilde{R}_{\varphi\theta}{}^{\varphi}{}_{\theta} = \tilde{R}_{\varphi\theta}{}^{\varphi}{}_{\theta} = -\tilde{R}_{\theta\varphi}{}^{\varphi}{}_{\theta} = -g^{\varphi\varphi}R_{\theta\varphi\varphi\theta}$$

$$= +g^{\varphi\varphi}\tilde{R}_{\theta\varphi\theta\varphi} = g^{\varphi\varphi}g_{\theta\theta}\tilde{R}_{\theta\varphi}{}^{\theta}{}_{\varphi} = \frac{1}{r^2\sin^2\theta}\cdot r^2\cdot\sin^2\theta = 1$$

$$\tilde{R}_{\theta\varphi} = \tilde{R}_{\alpha\theta}{}^{\alpha}{}_{\varphi} = \tilde{R}_{\theta\theta}{}^{\theta}{}_{\varphi} + \tilde{R}_{\varphi\theta}{}^{\varphi}{}_{\varphi} = 0$$

$$\tilde{R}_{\varphi\varphi} = \tilde{R}_{\alpha\varphi}{}^{\alpha}{}_{\varphi} = \tilde{R}_{\theta\varphi}{}^{\theta}{}_{\varphi} + \underbrace{\tilde{R}_{\varphi\varphi}{}^{\varphi}{}_{\varphi}}_{=0} = \sin^2\theta$$

$$\tilde{R} = g^{\mu\nu}\tilde{R}_{\mu\nu} = g^{\theta\theta}\tilde{R}_{\theta\theta} + \underbrace{g^{\theta\varphi}\tilde{R}_{\theta\varphi}}_{0} + \underbrace{g^{\varphi\theta}\tilde{R}_{\varphi\theta}}_{0} + g^{\varphi\varphi}\tilde{R}_{\varphi\varphi}$$

$$= \frac{1}{r^2} + \frac{1}{r^2\sin^2\theta}\sin^2\theta = \frac{2}{r^2}$$

$\longrightarrow$ final result: $\boxed{\tilde{R} = \frac{2}{r^2}}$

Scalar curvature of a sphere is given by its inverse radius squared!