6. The Levi-Civita connection

$$\partial_\mu \partial_\nu = \Gamma^\alpha{}_{\mu\nu}\, \partial_\alpha \quad \text{is not a unique equation for } \Gamma^\alpha{}_{\mu\nu}.$$

Many "connections" can satisfy this relation.

Reason: $\Gamma^\mu{}_{\nu\rho}$ is a connection $\Longrightarrow$ $\Gamma^\mu{}_{\nu\rho} + X^\mu{}_{\nu\rho}$ is a connection for any $\binom{1}{2}$ tensor $X^\mu{}_{\nu\rho}$.

How can we get a unique connection? Need to demand extra conditions!

1) Very common: "metric compatibility"
→ assume that covariant differentiation commutes with the metric

$$\nabla_\rho g_{\mu\nu} = 0$$

2) Vanishing torsion (this one is more ad hoc, cf. supergravity, Poincaré gauge gravity)

$$\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} = 0 = T_{\mu\nu}{}^\lambda$$

These two assumptions fix a unique connection. "Levi-Civita connection"
Turns out: Levi-Civita connection is given by $g_{\mu\nu}$ and its first derivatives!
Let us calculate it!

$$\nabla_\rho g_{\mu\nu} \equiv \partial_\rho g_{\mu\nu} - \Gamma^\lambda{}_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda{}_{\rho\nu} g_{\mu\lambda}$$

$$= \partial_\rho g_{\mu\nu} - \Gamma_{\nu\rho\mu} - \Gamma_{\mu\rho\nu} \overset{!}{=} 0$$

Recall from definition of $\Gamma^\lambda{}_{\mu\nu}$ that the λ-index is a vector index that we can raise and lower with $g_{\mu\nu}$ (unlike the other indices!)

Reorganize the terms:

$$\partial_\rho g_{\mu\nu} = \Gamma_{\nu\rho\mu} + \Gamma_{\mu\rho\nu} \qquad \text{(A)}$$

$$\partial_\mu g_{\rho\nu} = \Gamma_{\nu\mu\rho} + \Gamma_{\rho\mu\nu} \qquad \text{(B)} \qquad (\text{switch } \mu \leftrightarrow \rho)$$

$$\partial_\nu g_{\mu\rho} = \Gamma_{\rho\nu\mu} + \Gamma_{\mu\nu\rho} \qquad \text{(C)} \qquad (\text{switch } \nu \leftrightarrow \rho)$$

Add up in a certain way (tensorial equation, so it's OK and meaningful):

$$
\begin{aligned}
A + B - C &= \partial_\rho g_{\mu\nu} + \partial_\mu g_{\rho\nu} - \partial_\nu g_{\mu\rho} \\
&= \Gamma_{\nu\rho\mu} + \underline{\Gamma_{\mu\rho\nu}} + \Gamma_{\nu\mu\rho} + \underline{\underline{\Gamma_{\rho\mu\nu}}} - \underline{\underline{\Gamma_{\rho\nu\mu}}} - \underline{\Gamma_{\mu\nu\rho}} \\
&= \left(\Gamma_{\nu\rho\mu} + \Gamma_{\nu\mu\rho}\right) + \left(\Gamma_{\mu\rho\nu} - \Gamma_{\mu\nu\rho}\right) + \left(\Gamma_{\rho\mu\nu} - \Gamma_{\rho\nu\mu}\right) \\
&= \Gamma_{\nu\rho\mu} + \Gamma_{\nu\mu\rho} + T_{\rho\nu\mu} + T_{\mu\nu\rho} \\
&= \left(\Gamma_{\nu\rho\mu} + \cancel{\Gamma_{\nu\mu\rho}}\right) + \left(\Gamma_{\nu\rho\mu} - \cancel{\Gamma_{\nu\mu\rho}}\right) - \left(\Gamma_{\nu\rho\mu} - \Gamma_{\nu\mu\rho}\right) + T_{\rho\nu\mu} + T_{\mu\nu\rho} \\
&= 2\Gamma_{\nu\rho\mu} - T_{\rho\mu\nu} + T_{\rho\nu\mu} + T_{\mu\nu\rho}
\end{aligned}
$$

$$\longrightarrow \boxed{\Gamma_{\nu\rho\mu} = \frac{1}{2}\left(\partial_\rho g_{\mu\nu} + \partial_\mu g_{\rho\nu} - \partial_\nu g_{\mu\nu}\right) + \frac{1}{2}\left(T_{\rho\mu\nu} - T_{\rho\nu\mu} - T_{\mu\nu\rho}\right)}$$

metric-compatible connection <u>with</u> torsion

$$\longrightarrow \boxed{\begin{aligned} \tilde{\Gamma}_{\nu\rho\mu} &\equiv \frac{1}{2}\left(\partial_\rho g_{\mu\nu} + \partial_\mu g_{\rho\nu} - \partial_\nu g_{\mu\nu}\right) \\ \tilde{\Gamma}^\lambda{}_{\mu\nu} &\equiv \frac{1}{2} g^{\lambda\alpha}\left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right) \end{aligned}}$$

metric-compatible connection with torsion <u>set to zero</u>.

"Levi-Civita connection"