4. The metric tensor

Recall: tensor contractions are maps from tensors into the field (i.e. $\mathbb{R}$ or $\mathbb{C}$).

For example: $\underline{T} = T^{\mu}{}_{\nu}\, \partial_\mu \otimes dx^\nu$, $\quad \underline{v} = v^\mu \partial_\mu$, $\quad \underline{\omega} = \omega_\mu dx^\mu$

$$\underline{T}(\underline{\omega}, \underline{v}) = T^{\mu}{}_{\nu}\, \omega_\mu v^\nu \quad \text{is a number ("scalar")}$$

Tensor contractions are invariant under coordinate transformations.

$$T^{\mu}{}_{\nu}\, \omega_\mu v^\nu = \underbrace{\frac{\partial x^\mu}{\partial y^{\mu'}} \frac{\partial y^{\nu'}}{\partial x^\nu} T^{\mu'}{}_{\nu'}} \; \underbrace{\frac{\partial y^{\rho'}}{\partial x^\mu} \omega_{\rho'}} \; \underbrace{\frac{\partial x^\nu}{\partial y^{\sigma'}} v^{\sigma'}}$$

$$= \underbrace{\frac{\partial x^\mu}{\partial y^{\mu'}} \frac{\partial y^{\rho'}}{\partial x^\mu}}_{=\delta^{\rho'}_{\mu'}} \; \underbrace{\frac{\partial y^{\nu'}}{\partial x^\nu} \frac{\partial x^\nu}{\partial y^{\sigma'}}}_{=\delta^{\nu'}_{\sigma'}} T^{\mu'}{}_{\nu'}\, \omega_{\rho'} v^{\sigma'} = T^{\mu'}{}_{\nu'}\, \omega_{\mu'} v^{\nu'} \quad \checkmark$$

<u>But</u>: we don't know yet how to contract, say, a vector with itself.

Metric tensor $\underline{\underline{g}}$: a non-degenerate $\binom{0}{2}$ symmetric tensor that maps two vectors into a number. Coordinate basis: $\underline{\underline{g}} = g_{\mu\nu}\, dx^\mu \otimes dx^\nu$

Take two vectors $\underline{u}$ and $\underline{v}$:
$$g(\underline{u}, \underline{v}) = \underline{v} \curvearrowright [\, \underline{u} \curvearrowright \underline{\underline{g}}\,]$$
$$= \underline{v} \curvearrowright [\, u^\mu g_{\mu\nu}\, dx^\nu\,] = u^\mu v^\nu g_{\mu\nu}$$

Know that $u^\mu v^\nu g_{\mu\nu}$ is coordinate invariant. Also, u^μ transforms like a vector.

$\longrightarrow$ $v^\nu g_{\mu\nu}$ transforms as a co-vector!

<u>Notation</u>: "lowering an index"

$$v_\mu \equiv v^\nu g_{\mu\nu}$$

Metric is assumed to be non-degenerate. → inverse exists.

Inverse metric tensor $\underset{\sim}{g}^{-1}$: a non-degenerate $\binom{2}{0}$ symmetric tensor that maps two covectors into a number. $\underset{\sim}{g}^{-1} = g^{\mu\nu}\partial_\mu \otimes \partial_\nu$.

Relation to metric: $g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$.

Take two covectors $\underline{\omega}$ and $\underline{\lambda}$: $\underset{\sim}{g}^{-1}(\underline{\omega}, \underline{\lambda}) = \omega_\mu \lambda_\nu g^{\mu\nu}$

Know that $\omega_\mu \lambda_\nu g^{\mu\nu}$ is coordinate invariant. Also, ω_μ transforms like a covector.

→ $\lambda_\nu g^{\mu\nu}$ transforms like a vector!

<u>Notation:</u> "raising an index"

$$\lambda^\mu \equiv \lambda_\nu g^{\mu\nu}$$

<u>Examples:</u> $g^{\mu\nu} g^{\rho\sigma} T_{\mu\rho} = T^{\nu\sigma}$

$$g^{\mu\alpha} T_{\rho\sigma\nu\alpha\beta\omega\lambda} = T_{\rho\sigma\nu}{}^{\mu}{}_{\beta\omega\lambda}$$

<u>Norm of a tensor:</u> we can use the metric and inverse metric to define the norm (or magnitude) of a tensor $\underset{\sim}{T}$.

$$\underset{\sim}{T} = T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \ldots \nu_q}\, \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_p} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_q}$$

$$|\underset{\sim}{T}|^2 \equiv T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \ldots \nu_q}\, g_{\mu_1 \rho_1} \cdots g_{\mu_p \rho_p}\, g^{\nu_1 \sigma_1} \cdots g^{\nu_q \sigma_q}\, T^{\rho_1 \cdots \rho_p}{}_{\sigma_1 \ldots \sigma_q}$$

$$= \underbrace{T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \ldots \nu_q}}_{\binom{p}{q}} \underbrace{T_{\mu_1 \cdots \mu_p}{}^{\nu_1 \ldots \nu_q}}_{\binom{q}{p}}$$

Simple example: metric in $\mathbb{R}^3$

$$\underset{\sim}{g} = dx \otimes dx + dy \otimes dy + dz \otimes dz$$

What's the norm of the vector $\underline{v} = x^\mu \partial_\mu$?

$$g(\underline{v}, \underline{v}) = g_{\mu\nu}\, x^\mu x^\nu = x^2 + y^2 + z^2 = r^2 = |\underline{v}|^2 .$$

$\rightarrow$ unit vector $\underline{u} = \frac{1}{r} x^\mu \partial_\mu = \frac{1}{r}\left(x\partial_x + y\partial_y + z\partial_z \right)$