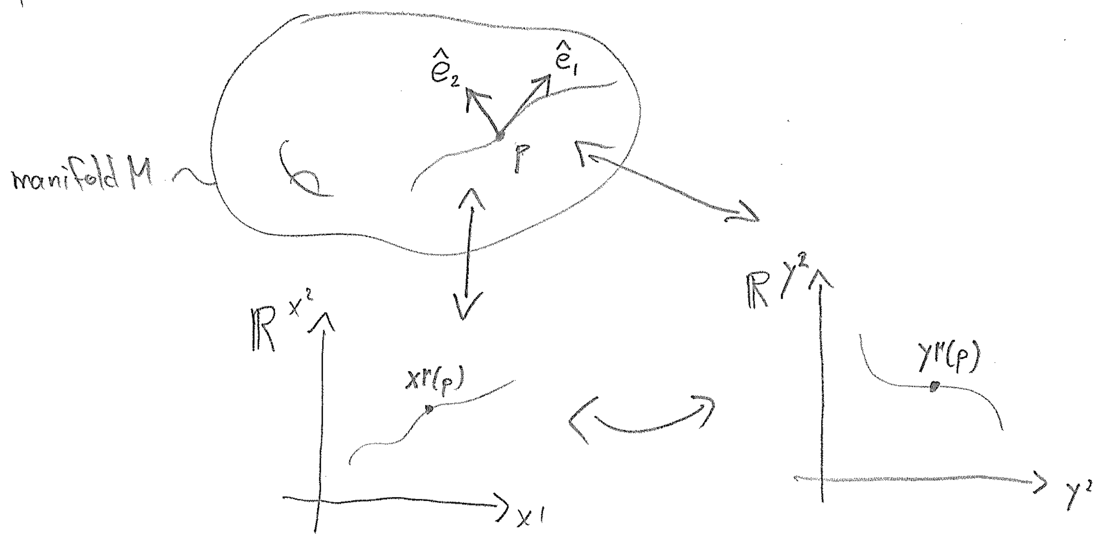
Big picture:



A point $p \in M$ has coordinates $x^\mu(p)$ = set of $\mathbb{R}$-numbers.
$y^\mu(p)$ = set of different $\mathbb{R}$-numbers. } x^μ, y^μ different coords.

E.g. $x^\mu = (x, y, z)$, $y^\mu = (r, \theta, \varphi)$. Not a vector, points.

Idea: expand abstract (co-)basis into coordinates!

$$\hat{e}_i = e^\alpha{}_i \underbrace{\frac{\partial}{\partial x^\alpha}}_{\text{coord. basis}}, \qquad \hat{\vartheta}^j = e_\alpha{}^j \underbrace{dx^\alpha}_{\text{coord. cobasis}}$$

why do we use these symbols? see below!

$$\delta_i^j = \hat{e}_i \lrcorner \hat{\vartheta}^j \equiv e^\alpha{}_i \frac{\partial}{\partial x^\alpha} \lrcorner e_\beta{}^j dx^\beta$$

$$= e^\alpha{}_i e_\beta{}^j \frac{\partial}{\partial x^\alpha} \lrcorner dx^\beta = e^\alpha{}_i e_\beta{}^j \boxed{\frac{\partial x^\beta}{\partial x^\alpha}} = \underbrace{e^\alpha{}_i}_{\text{"vielbein"}} e_\alpha{}^j$$

$$\frac{\partial x^\beta}{\partial x^\alpha} = \delta_\alpha^\beta$$

$\longrightarrow$ $e^\alpha{}_i$ and $e_\beta{}^j$ are inverse to each other.

$$\underline{T} = T^i{}_j \, \hat{e}_i \otimes \hat{\vartheta}^j = T^i{}_j \underbrace{\delta_i^k}_{= e^\alpha{}_i e_\alpha{}^k} \hat{e}_k \otimes \underbrace{\delta_\ell^j}_{= e^\beta{}_\ell e_\beta{}^j} \hat{\vartheta}^\ell = \boxed{T^i{}_j e^\alpha{}_i e_\beta{}^j} \partial_\alpha \otimes dx^\beta$$

$$= \boxed{T^\alpha{}_\beta} \partial_\alpha \otimes dx^\beta$$

General tensor can be expanded <u>either</u> in abstract basis $\{\hat{e}_i, \hat{\vartheta}^j\}$
<u>or</u> in coordinate basis $\{\partial_\mu, dx^\nu\}$.

$$\underline{T} = T^{ij\ldots}{}_{k\ell\ldots}\, \hat{e}_i \otimes \hat{e}_j \otimes \ldots \otimes \hat{\vartheta}^k \otimes \hat{\vartheta}^\ell \otimes \ldots$$
$$= \underbrace{T^{\alpha\beta\ldots}{}_{\mu\nu\ldots}}\, \partial_\alpha \otimes \partial_\beta \otimes \ldots \otimes dx^\mu \otimes dx^\nu \otimes \ldots$$

Latin indices = abstract indices,
Greek indices = coordinate indices.

$\rightarrow$ relation of components
$$T^{ij}{}_{k\ell} = e_\alpha{}^i\, e_\beta{}^j\, e^\mu{}_k\, e^\nu{}_\ell\, T^{\alpha\beta}{}_{\mu\nu}$$
$$T^{\alpha\beta}{}_{\mu\nu} = e^\alpha{}_i\, e^\beta{}_j\, e_\mu{}^k\, e_\nu{}^\ell\, T^{ij}{}_{k\ell}$$

<u>For now, let us work in a coordinate basis!</u> (Will get back to general basis later!)

Examples of tensors: $\underline{v} = x\partial_x + y\partial_y$, $\quad \underline{\omega} = dy - dx$,
$$\underline{T} = zx^2\,\partial_x \otimes dz - y\,\partial_y \otimes dy$$

Contractions:
$$\underline{\omega}(\underline{v}) = (x\partial_x + y\partial_y) \curvearrowright (dy - dx)$$
$$= \underbrace{x\partial_x \curvearrowright dy}_{=0} - \underbrace{x\partial_x \curvearrowright dx}_{=1} + \underbrace{y\partial_y \curvearrowright dy}_{1} - \underbrace{y\partial_y \curvearrowright dx}_{=0}$$
$$= y - x.$$

New coordinates: $y^\mu = y^\mu(x) \longrightarrow \dfrac{\partial y^\mu}{\partial x^\nu} = \text{Jacobian}$

$$\frac{\partial}{\partial x^\mu} \rightarrow \frac{\partial y^{\nu'}}{\partial x^\mu}\frac{\partial}{\partial y^{\nu'}}, \qquad dx^\mu \rightarrow \frac{\partial x^\mu}{\partial y^{\nu'}}\, dy^{\nu'}$$

But: tensors have to be invariant under change of coordinates!

$$\underline{T} = T^\mu{}_\nu\, \partial_\mu \otimes dx^\nu = T^{\mu'}{}_{\nu'}\, \partial_{\mu'} \otimes dy^{\nu'}$$
$$= T^{\mu'}{}_{\nu'}\, \frac{\partial}{\partial y^{\mu'}} \otimes dy^{\nu'} = \underbrace{T^{\mu'}{}_{\nu'}\, \frac{\partial x^\mu}{\partial y^{\mu'}}}\, \frac{\partial}{\partial x^\mu} \otimes \underbrace{\frac{\partial y^{\nu'}}{\partial x^\nu}}\, dx^\nu$$
$$= \boxed{T^{\mu'}{}_{\nu'}\, \frac{\partial x^\mu}{\partial y^{\mu'}}\, \frac{\partial y^{\nu'}}{\partial x^\nu}}\, \partial_\mu \otimes dx^\nu \equiv \boxed{T^\mu{}_\nu}\, \partial_\mu \otimes dx^\nu$$

$$\longrightarrow T^\mu{}_\nu = \frac{\partial x^\mu}{\partial y^{\mu'}}\, \frac{\partial y^{\nu'}}{\partial x^\nu}\, T^{\mu'}{}_{\nu'} \qquad \text{transformation law of tensor components.}$$

But it is not always so complicated! Example:

$\underline{\eta} = -dt \otimes dt + dx \otimes dx \qquad (t,x)\text{-coordinates}$

$\eta_{tt} = -1, \quad \eta_{xx} = +1.$

New coordinates: $u = t - x, \quad v = t + x$

$\longrightarrow t = \frac{u+v}{2}, \quad x = \frac{v-u}{2}$

$\longrightarrow dt = \frac{1}{2}(du + dv), \quad dx = \frac{1}{2}(dv - du).$

$$\Rightarrow \underline{\eta} = -\left[\tfrac{1}{2}(du+dv)\right] \otimes \left[\tfrac{1}{2}(du+dv)\right] + \left[\tfrac{1}{2}(dv-du)\right] \otimes \left[\tfrac{1}{2}(dv-du)\right]$$

$$= -\frac{1}{4}(du+dv)\otimes(du+dv) + \frac{1}{4}(dv-du)\otimes(dv-du)$$

$$= -\frac{1}{4}\Big[\cancel{du\otimes du} + du\otimes dv + dv\otimes du + \cancel{dv\otimes dv}$$

$$-\cancel{dv\otimes dv} + du\otimes dv + dv\otimes du - \cancel{du\otimes du}\Big] = -\frac{du\otimes dv + dv\otimes du}{2}$$

Read off components: $\eta_{uv} = \eta_{vu} = -\frac{1}{2}$, rest zero.

$\underline{\text{Alternative}}$: $\eta_{uv} = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v}\eta_{xx} + \frac{\partial x}{\partial v}\frac{\partial x}{\partial u}\eta_{xx}$

$$= \left(-\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\cdot 1 + \left(\tfrac{1}{2}\right)\cdot\left(-\tfrac{1}{2}\right)\cdot 1 = -\frac{1}{2} \quad \checkmark$$