How to calculate the Hodge dual of a p-form?

1. In what dimension are we? $\rightarrow n$
2. What are the coordinates? E.g. $x^\mu = (x, y, z)$ or $y^{\mu'} = (t, r, \varphi)$
3. Agree on an "orientation" so we can define the **Levi-Civita symbol $\epsilon_{\mu_1 \ldots \mu_n}$**

   $\epsilon_{xyz} \equiv +1, \quad \epsilon_{tr\varphi} = +1$ (conventional choice)

   $\epsilon_{\mu_1 \ldots \mu_n}$ is totally antisymmetric symbol.

4. Given the metric, calculate $\sqrt{|\det g_{\mu\nu}|}$.

   Then we can define the Levi-Civita tensor $\varepsilon_{\mu_1 \ldots \mu_n} \equiv \sqrt{|\det g_{\mu\nu}|}\, \epsilon_{\mu_1 \ldots \mu_n}$.

5. Take your p-form $\omega_{\mu_1 \ldots \mu_p}$. The dual is

$$\ast\underline{\omega} = \frac{1}{p!(n-p)!}\, \omega^{\mu_1 \ldots \mu_p}\, \varepsilon_{\mu_1 \ldots \mu_p \mu_{p+1} \ldots \mu_n}\, dx^{\mu_{p+1}} \wedge \ldots \wedge dx^{\mu_n}.$$

$$(\ast\omega)_{\mu_1 \ldots \mu_{n-p}} = \frac{1}{p!(n-p)!}\, \omega^{\nu_1 \ldots \nu_p}\, \varepsilon_{\nu_1 \ldots \nu_p \mu_1 \ldots \mu_{n-p}}$$

(Note: $\underline{\omega} = \frac{1}{p!}\, \omega_{\mu_1 \ldots \mu_p}\, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}$)

10. The codifferential

Recall: d = exterior derivative = maps p-form to (p+1)-form

Now: δ = codifferential = maps p-form to (p−1)-form

$$\delta\underline{\omega} \equiv \ast d \ast \underline{\omega}$$

- $\underline{\omega}$: p
- $\ast\underline{\omega}$: $n-p$
- $d\ast\underline{\omega}$: $n-p+1$
- $\ast d\ast\underline{\omega}$: $n-(n-p+1) = p-1$

$$\delta^2\underline{\omega} = \ast d \ast \ast d \ast \underline{\omega} = (-1)^{(n-p+1)(p-1)+s} \ast d d \ast \omega = 0$$

But: no Leibniz rule!

Can show: $\delta\underline{\omega} = (\text{some factor}) \times (\partial^\rho \omega_{\rho\alpha_1 \ldots \alpha_{p-1}})\, dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_{p-1}}$

Why is it useful? → E.g. Maxwell equations

Homework: $d\underline{F} = 0$ means $\vec{\nabla}\cdot\vec{B} = 0$ and $\vec{\nabla}\times\vec{E} = -\partial_t \vec{B}$

Today: $\delta \underline{F} = \underline{j}$ means $\vec{\nabla}\cdot\vec{E} = \rho$ and $\vec{\nabla}\times\vec{B} = \partial_t \vec{E} + \vec{j}$

($\delta\underline{F}$: 2-form; $\underline{j}$: 1-form)

Check: $\underline{j} = \rho\, dt + j_x\, dx + j_y\, dy + j_z\, dz$

$\underline{F} = (E_x\, dx + E_y\, dy + E_z\, dz)\wedge dt + B_x\, dy\wedge dz + B_y\, dz\wedge dx + B_z\, dx\wedge dy$

Question: what is $\delta \underline{F} = *d*\underline{F}$?

$$*\underline{F} = E_x *(dx\wedge dt) + E_y(*dy\wedge dt) + E_z *(dz\wedge dt) + B_x *(dy\wedge dz) + B_y *(dz\wedge dx) + B_z *(dx\wedge dy)$$

use: $*(dx\wedge dt) = *\underline{\omega}$ with $p=2,\ n=4,\ \omega_{xt} = -\omega_{tx} = 1$

$$= \frac{1}{2!}\frac{1}{(4-2)!}\,\omega^{\mu\nu}\varepsilon_{\mu\nu\rho\sigma}\,dx^\rho\wedge dx^\sigma = \frac{1}{4}\left(\omega^{tx}\varepsilon_{tx\rho\sigma} + \omega^{xt}\varepsilon_{xt\rho\sigma}\right)dx^\rho\wedge dx^\sigma$$

$$= \frac{1}{4}\left(1\cdot\varepsilon_{txyz} + 1\cdot\varepsilon_{txzy}(-1) + (-1)\,\varepsilon_{xtyz} + (-1)\,\varepsilon_{xtzy}(-1)\right) dy\wedge dz$$

$$= dy\wedge dz\ ,\quad \text{and so on}$$

$$= E_x\, dy\wedge dz + E_y\, dz\wedge dx + E_z\, dx\wedge dy - (B_x\, dx + B_y\, dy + B_z\, dz)\wedge dt$$

$$d*F = \partial_t E_x\, dt\wedge dy\wedge dz + \partial_x E_x\, dx\wedge dy\wedge dz + \dots - \partial_y B_x\, dy\wedge dx\wedge dt - \partial_z B_x\, dz\wedge dx\wedge dt - \dots$$

$$*d*F = -(\partial_t E_x)\, dx + (\partial_x E_x)\, dt - (\partial_y B_x)\, dz + (\partial_z B_x)\, dy + \dots$$

$$= \rho\, dt + j_x\, dx + j_y\, dy + j_z\, dz$$

→ $\boxed{\vec{\nabla}\cdot\vec{E} = \rho\ ,\quad \vec{\nabla}\times\vec{B} = \partial_t\vec{E} + \vec{j}}$ full proof: homework!