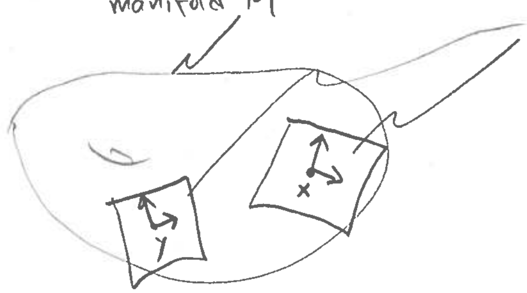
## 1. Vectors, covectors, matrices, and tensors

Big picture:




local "tangent spaces"

in each tangent space: linear algebra ← today

connecting tangent spaces: differential geometry

Goal for today: understand abstract tensors and their algebra.

Let's work in n dimensions. Basis $\hat{e}_i$, cobasis $\hat{\vartheta}^j$. $i, j, k = 1, \ldots, n$ indices.
They are fundamental objects, decorated with hats. All tensors can be expanded in this basis.

vector: $\underline{v} = v^i \hat{e}_i$

covector: $\underline{\omega} = \omega_i \hat{\vartheta}^i$

matrix: $\underline{M} = M_i{}^j \hat{\vartheta}^i \otimes \hat{e}_j$

tensor: $\underline{T} = T^{j_1 j_2 \cdots j_p}{}_{i_1 i_2 \cdots i_q}\, \hat{e}_{j_1} \otimes \hat{e}_{j_2} \otimes \cdots \otimes \hat{e}_{j_p} \otimes \hat{\vartheta}^{i_1} \otimes \hat{\vartheta}^{i_2} \otimes \cdots \otimes \hat{\vartheta}^{i_q}$ "tensor product"

rank $\binom{p}{q}$ tensor

Basis and cobasis are <u>dual</u> to one another. We <u>assume</u> they are related via

$$\hat{e}_i \lrcorner \hat{\vartheta}^j = \delta_i^j = \begin{cases} 1 : & i = j \\ 0 : & \text{else} \end{cases}$$

"hook"

"interior product" ($\neq$ "innerproduct"!)

This relation is very similar in spirit to $\underbrace{\langle \phi_{\vec{k}}|}_{\text{covector}}\underbrace{|\psi_{\vec{k}'}\rangle}_{\text{vector}} = \delta(\vec{k} - \vec{k}')$ in QM.

The "tensor product" is just our way to collect basis and cobasis elements.
Think of $|\psi\rangle \otimes |\chi\rangle \otimes \ldots \otimes |\phi\rangle$ in quantum mechanics.
Order matters, though. $\hat{e}_i \otimes \hat{e}_j \neq \hat{e}_j \otimes \hat{e}_i$, and $\hat{e}_i \otimes \hat{\vartheta}^j \neq \hat{\vartheta}^j \otimes \hat{e}_i$.
Will become more clear when we work with them, just a minute.

Tensors form a "vector space."

(More precisely, tensors of the same rank.) Basic properties:

- Can add tensors of same rank, get another tensor of same rank.

$\underline{A}$, $\underline{B}$ rank $\binom{0}{p}$ tensors. $\underline{A} + \underline{B} = A_{i_1 \ldots i_p} \hat{\vartheta}^{i_1} \otimes \ldots \otimes \hat{\vartheta}^{i_p} + B_{i_1 \ldots i_p} \hat{\vartheta}^{i_1} \otimes \ldots \otimes \hat{\vartheta}^{i_p}$

$$= \left( A_{i_1 \ldots i_p} + B_{i_1 \ldots i_p} \right) \hat{\vartheta}^{i_1} \otimes \ldots \otimes \hat{\vartheta}^{i_p}$$

- Can multiply tensors with elements of the field (here: $\mathbb{R}$, sometimes: $\mathbb{C}$) and still have a tensor of same rank. $\underline{A} \to 3 \times \underline{A}$, etc.

We call the space of rank $\binom{p}{q}$ tensors $V^p_q$. $\underline{T} \in V^p_q \to T_{i_1 \ldots i_q}{}^{j_1 \ldots j_p}$ components

$\#(\underline{T})$ = number of independent components $= n^{p+q}$.

Tensor contractions: use: $\hat{e}_i \lrcorner \hat{\vartheta}^j = \delta_i^j$

$$\underline{v} \lrcorner \underline{w} = (v^i \hat{e}_i) \lrcorner (w_j \hat{\vartheta}^j) = v^i w_j \; \hat{e}_i \lrcorner \hat{\vartheta}^j = v^i w_j \delta_i^j = v^i w_i$$

We also write this as $\underline{w}(\underline{v})$. Or as $\underline{v}(\underline{w})$. Depends on your point of view.

$$\underline{M}(\underline{v}, \underline{w}) \equiv M_i{}^j v^i w_j$$

$$\underline{T}(\underbrace{\underline{v}, \underline{v}, \ldots, \underline{v}}_{q}, \underbrace{\underline{w}, \underline{w}, \ldots, \underline{w}}_{p}) \equiv T^{j_1 j_2 \ldots j_p}{}_{i_1 i_2 \ldots i_q} v^{i_1} v^{i_2} \ldots v^{i_q} w_{j_1} w_{j_2} \ldots w_{j_p}$$

A tensor of rank $\binom{p}{q}$ maps $p$ covectors and $q$ vectors into a <u>scalar</u>.

Apply $q$ vectors (left to right) and $p$ covectors (left to right). Each slot = 1 index.

Partial contraction: $\underline{M}(\underline{v}, \bullet) = M_i{}^j v^i \hat{e}_j$

$$\underline{M}(\bullet, \underline{w}) = M_i{}^j w_j \hat{\vartheta}^i$$