Issued: December 13, 2021
Due: 12pm, December 15, 2021
Official website: http://spintwo.net/Courses/PHYS-581-Differential-Geometry-for-Physicists/

You are expected to work on this exam on your own; complete it with the lecture notes and no other external help. If you have questions you can email the instructor, Jens Boos (jboos@wm.edu). Please *do not* discuss details of this exam with other students before submitting your work.

# 1 Coordinate transformations

Consider two-dimensional Minkowski spacetime with the metric

$$\underline{g} = g_{\mu\nu}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = -\mathrm{d}t \otimes \mathrm{d}t + \mathrm{d}x \otimes \mathrm{d}x\,, \tag{1}$$

and the vector fields $\underline{\xi} = \partial_t$, $\underline{\zeta} = \partial_x$, and $\underline{\eta} = x\partial_t + t\partial_x$.

(a) Read off the components $\xi^\mu$, $\zeta^\mu$, and $\eta^\mu$.

(b) Calculate the norm of $\underline{\xi}$, given by $|\underline{\xi}|^2 = g_{\mu\nu}\xi^\mu\xi^\nu$, and the norm of $\underline{\zeta}$, given by $|\underline{\zeta}|^2 = g_{\mu\nu}\zeta^\mu\zeta^\nu$.

(c) Calculate the norm of $\underline{\eta}$ given by $|\underline{\eta}|^2 = g_{\mu\nu}\eta^\mu\eta^\nu$. Where does it vanish?

(d) Plot the vector fields $\underline{\xi}$, $\underline{\zeta}$, and $\underline{\eta}$ in the $tx$-plane.

Consider now new coordinates $y^{\mu'} = (u, v)$ given by

$$u = \frac{t-x}{\sqrt{2}}\,, \quad v = \frac{t+x}{\sqrt{2}}\,. \tag{2}$$

(e) Show that the metric in these coordinates takes the form

$$\underline{g} = g_{\mu'\nu'}\,\mathrm{d}y^{\mu'} \otimes \mathrm{d}y^{\nu'} = -\mathrm{d}u \otimes \mathrm{d}v - \mathrm{d}v \otimes \mathrm{d}u\,. \tag{3}$$

(f) Show that in the new coordinates one has

$$\underline{\xi} = \frac{1}{\sqrt{2}}\left(\partial_v + \partial_u\right)\,, \quad \underline{\zeta} = \frac{1}{\sqrt{2}}\left(\partial_v - \partial_u\right)\,, \quad \underline{\eta} = v\partial_v - u\partial_u\,. \tag{4}$$

## 2 Scalar curvature

Consider the following metric:

$$\underline{g} = g_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu = \rho^2 \mathrm{d}\theta \otimes \mathrm{d}\theta + \rho^2 \sinh^2\theta\, \mathrm{d}\varphi \otimes \mathrm{d}\varphi\,, \tag{5}$$

where $\rho$ is a constant.

(a) Give the metric components $g_{\theta\theta}$, $g_{\theta\varphi}$, and $g_{\varphi\varphi}$.

(b) Give the inverse metric components $g^{\theta\theta}$, $g^{\theta\varphi}$, and $g^{\varphi\varphi}$.

(c) The Levi–Civita connection is given by

$$\tilde{\Gamma}^\mu{}_{\nu\rho} = \frac{1}{2} g^{\mu\alpha} \left(\partial_\nu g_{\alpha\rho} + \partial_\rho g_{\alpha\nu} - \partial_\alpha g_{\nu\rho}\right)\,, \tag{6}$$

and recall that $\tilde{\Gamma}^\mu{}_{\nu\rho} = \tilde{\Gamma}^\mu{}_{\rho\nu}$. For the metric (5) show that

$$\tilde{\Gamma}^\theta{}_{\theta\theta} = 0\,, \tag{7}$$

$$\tilde{\Gamma}^\theta{}_{\theta\varphi} = 0\,, \tag{8}$$

$$\tilde{\Gamma}^\theta{}_{\varphi\varphi} = -\sinh\theta\cosh\theta\,, \tag{9}$$

$$\tilde{\Gamma}^\varphi{}_{\theta\theta} = 0\,, \tag{10}$$

$$\tilde{\Gamma}^\varphi{}_{\theta\varphi} = \frac{\cosh\theta}{\sinh\theta}\,, \tag{11}$$

$$\tilde{\Gamma}^\varphi{}_{\varphi\varphi} = 0\,. \tag{12}$$

(d) The Riemann curvature tensor $\tilde{R}_{\alpha\beta}{}^\mu{}_\nu$ is given by

$$\tilde{R}_{\alpha\beta}{}^\mu{}_\nu = \partial_\alpha \tilde{\Gamma}^\mu{}_{\beta\nu} - \partial_\beta \tilde{\Gamma}^\mu{}_{\alpha\nu} + \tilde{\Gamma}^\mu{}_{\alpha\lambda}\tilde{\Gamma}^\lambda{}_{\beta\nu} - \tilde{\Gamma}^\mu{}_{\beta\lambda}\tilde{\Gamma}^\lambda{}_{\alpha\nu}\,. \tag{13}$$

Show that

$$\tilde{R}_{\theta\varphi}{}^\theta{}_\varphi = -\sinh^2\theta\,, \qquad \tilde{R}_{\varphi\theta}{}^\varphi{}_\theta = -1\,. \tag{14}$$

(e) The Ricci curvature tensor is

$$\tilde{R}_{\mu\nu} = \tilde{R}_{\alpha\mu}{}^\alpha{}_\nu\,. \tag{15}$$

Show that

$$\tilde{R}_{\theta\theta} = -1\,, \qquad \tilde{R}_{\varphi\varphi} = -\sinh^2\theta\,, \qquad \tilde{R}_{\theta\varphi} = \tilde{R}_{\varphi\theta} = 0\,. \tag{16}$$

(f) Last, recall that the Ricci scalar is $\tilde{R} = g^{\mu\nu}\tilde{R}_{\mu\nu}$. Show that

$$\tilde{R} = -\frac{2}{\rho^2}\,. \tag{17}$$

(g) This means that this space is one of constant *negative* curvature. Compare this to the metric of the sphere. What is the difference?

# 3 Differential forms in three dimensions

It is our goal to derive an expression for the so-called *vector Laplacian*

$$\vec{\nabla}^2 \vec{A} \equiv \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{A})\,. \tag{18}$$

To that end, consider the three-dimensional metric $\underline{g}$ and the 1-form $\underline{A}$ given by

$$\begin{aligned} \underline{g} &= \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}z \otimes \mathrm{d}z\,, \\ \underline{A} &= A_x \mathrm{d}x + A_y \mathrm{d}y + A_z \mathrm{d}z\,. \end{aligned} \tag{19}$$

In a previous assignment we have shown

$$\begin{aligned} &\star\,\mathrm{d}x = \mathrm{d}y \wedge \mathrm{d}z\,, \quad \star\mathrm{d}y = \mathrm{d}z \wedge \mathrm{d}x\,, \quad \star\mathrm{d}z = \mathrm{d}x \wedge \mathrm{d}y\,, \\ &\star\,(\mathrm{d}x \wedge \mathrm{d}y) = \mathrm{d}z\,, \quad \star\,(\mathrm{d}y \wedge \mathrm{d}z) = \mathrm{d}x\,, \quad \star\,(\mathrm{d}z \wedge \mathrm{d}x) = \mathrm{d}y\,, \\ &\star\,1 = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\,, \quad \star\,(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z) = 1\,, \end{aligned} \tag{20}$$

and in here you can make use of these identities freely and you do *not* need to de-derive them.

(a) Compute $\mathrm{d} \star \mathrm{d} \star \underline{A}$ and show that it is given by

$$\begin{aligned} \mathrm{d} \star \mathrm{d} \star \underline{A} = \;& (\partial_x \partial_x A_x)\mathrm{d}x + (\partial_x \partial_y A_x)\mathrm{d}y + (\partial_x \partial_z A_x)\mathrm{d}z \\ &+ (\partial_y \partial_x A_y)\mathrm{d}x + (\partial_y \partial_y A_y)\mathrm{d}y + (\partial_y \partial_z A_y)\mathrm{d}z \\ &+ (\partial_z \partial_x A_z)\mathrm{d}x + (\partial_z \partial_y A_z)\mathrm{d}y + (\partial_z \partial_z A_z)\mathrm{d}z\,. \end{aligned} \tag{21}$$

(b) Argue that $\mathrm{d} \star \mathrm{d} \star \underline{A}$ has the same structure as $\vec{\nabla}(\vec{\nabla} \cdot \vec{A})$. No rigorous proof necessary! *Hint:* Think of the order of operations as consisting of two steps, $\mathrm{d}(\star \mathrm{d} \star \underline{A})$.

(c) Compute $\star \mathrm{d} \star \mathrm{d}\underline{A}$ and show that it is given by

$$\begin{aligned} \star \mathrm{d} \star \mathrm{d}\underline{A} = &- (\partial_y \partial_y A_x)\mathrm{d}x - (\partial_z \partial_z A_x)\mathrm{d}x + (\partial_x \partial_y A_x)\mathrm{d}y + (\partial_z \partial_x A_x)\mathrm{d}z \\ &+ (\partial_x \partial_y A_y)\mathrm{d}x - (\partial_x \partial_x A_y)\mathrm{d}y - (\partial_z \partial_z A_y)\mathrm{d}y + (\partial_z \partial_y A_y)\mathrm{d}z \\ &+ (\partial_x \partial_z A_z)\mathrm{d}x + (\partial_y \partial_z A_z)\mathrm{d}y - (\partial_x \partial_x A_z)\mathrm{d}z - (\partial_y \partial_y A_z)\mathrm{d}z\,. \end{aligned} \tag{22}$$

(d) Argue that $\star \mathrm{d} \star \mathrm{d}\underline{A}$ has the same structure as $\vec{\nabla} \times (\vec{\nabla} \times \vec{A})$. No rigorous proof necessary! *Hint:* Think of the order of operations as consisting of two steps, $\star \mathrm{d}(\star \mathrm{d}\underline{A})$.

(e) By using the previous two results, show that

$$\mathrm{d} \star \mathrm{d} \star \underline{A} \;-\; \star \mathrm{d} \star \mathrm{d}\underline{A} = (\partial_x \partial_x + \partial_y \partial_y + \partial_z \partial_z)\underline{A}\,. \tag{23}$$

The right-hand side is the analog of the vector Laplacian in Eq. (18), as applied to a 1-form in Euclidean coordinates, so together with our arguments in (b) and (d) we have proven Eq. (18)!

# 4 Differential forms in four dimensions

Consider four-dimensional Minkowski spacetime with the metric

$$\underline{g} = g_{\mu\nu}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = -\mathrm{d}t \otimes \mathrm{d}t + \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}z \otimes \mathrm{d}z\,. \tag{24}$$

In previous assignments, we have shown the following relations (with the orientation $\varepsilon_{txyz} = +1$):

$$\begin{aligned} &\star(\mathrm{d}x \wedge \mathrm{d}t) = \mathrm{d}y \wedge \mathrm{d}z\,, \quad \star(\mathrm{d}y \wedge \mathrm{d}t) = \mathrm{d}z \wedge \mathrm{d}x\,, \quad \star(\mathrm{d}z \wedge \mathrm{d}t) = \mathrm{d}x \wedge \mathrm{d}y\,, \\ &\star(\mathrm{d}x \wedge \mathrm{d}y) = \mathrm{d}t \wedge \mathrm{d}z\,, \quad \star(\mathrm{d}y \wedge \mathrm{d}z) = \mathrm{d}t \wedge \mathrm{d}x\,, \quad \star(\mathrm{d}z \wedge \mathrm{d}x) = \mathrm{d}t \wedge \mathrm{d}y\,. \end{aligned} \tag{25}$$

Also, recall the electromagnetic field strength 2-form

$$\underline{F} = E_x\,\mathrm{d}x \wedge \mathrm{d}t + E_y\,\mathrm{d}y \wedge \mathrm{d}t + E_z\,\mathrm{d}z \wedge \mathrm{d}t + B_x\,\mathrm{d}y \wedge \mathrm{d}z + B_y\,\mathrm{d}z \wedge \mathrm{d}x + B_z\,\mathrm{d}x \wedge \mathrm{d}y\,, \tag{26}$$

as well as its dual

$$\star\underline{F} = E_x\,\mathrm{d}y \wedge \mathrm{d}z + E_y\,\mathrm{d}z \wedge \mathrm{d}x + E_z\,\mathrm{d}x \wedge \mathrm{d}y - B_x\,\mathrm{d}x \wedge \mathrm{d}t - B_y\,\mathrm{d}y \wedge \mathrm{d}t - B_z\,\mathrm{d}z \wedge \mathrm{d}t\,. \tag{27}$$

You may use all of the above expressions freely in this exercise and you do not need to re-derive any of them.

(a) Compute $\underline{F} \wedge \star\underline{F}$ and show that it is given by

$$\underline{F} \wedge \star\underline{F} = (\vec{B}^2 - \vec{E}^2)\,\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\,, \tag{28}$$

where we defined $\vec{E}^2 \equiv E_x^2 + E_y^2 + E_z^2$ and $\vec{B}^2 \equiv B_x^2 + B_y^2 + B_z^2$ for convenience. *Hint:* You do *not* have to write out all 36 terms of this product. It is entirely sufficient if you only write down the non-vanishing terms, and argue that the others indeed vanish.

(b) Compute $\underline{F} \wedge \underline{F}$ and show that it is given by

$$\underline{F} \wedge \underline{F} = -2\,\vec{E} \cdot \vec{B}\,\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\,, \tag{29}$$

where we defined $\vec{E} \cdot \vec{B} \equiv E_x B_x + E_y B_y + E_z B_z$ for convenience. *Hint:* You do *not* have to write out all 36 terms of this product. It is entirely sufficient if you only write down the non-vanishing terms, and argue that the others indeed vanish.

(c) $\underline{F} \wedge \star\underline{F}$ can be calculated in any number of spacetime dimensions $n$ (in the above, $n = 4$). However, $\underline{F} \wedge \underline{F}$ is trivial if $n \leq 3$. Why? *Hint:* Think about how many independent components a $p$-form has in $n$-dimensions for $p = 4$ and different choices of $n$.

(d) Recall $\underline{F} = \mathrm{d}\underline{A}$ and the homogeneous Maxwell equations $\mathrm{d}\underline{F} = 0$. Using this, prove that $\underline{F} \wedge \underline{F} = \mathrm{d}(\underline{F} \wedge \underline{A})$, *i.e.*, that it is a total derivative.