Issued: November 19, 2021
Due: 11am, Dec 3, 2021
Official website: http://spintwo.net/Courses/PHYS-581-Differential-Geometry-for-Physicists/

Please work on this problem set on your own; it should be possible to complete it with the lecture notes and no other external help. If you have questions you can email the instructor, Jens Boos (jboos@wm.edu), or make use of the office hours on Monday, 10am–11am, Small 235. *After* you have completed the assignment please feel free to discuss it with other students.

1 Hodge dual

In this exercise it is our goal to demystify the Hodge dual a little bit, by computing it in some simple examples. Consider a p-form $\underline{\omega}$, which has the expansion

$$\underline{\omega} = \frac{1}{p!}\omega_{\mu_1 \dots \mu_p} \, \mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_p} \, . \tag{1}$$

Its Hodge dual in n dimensions is defined as

$$\star\underline{\omega} \equiv \frac{1}{p!(n-p)!}\omega^{\mu_1 \dots \mu_p}\varepsilon_{\mu_1 \dots \mu_p \, \mu_{p+1} \dots \mu_n} \, \mathrm{d}x^{\mu_{p+1}} \wedge \dots \wedge \mathrm{d}x^{\mu_n} \, . \tag{2}$$

For this first part, assume that we are in three dimensions ($n = 3$) with the metric $\underline{g} = \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}z \otimes \mathrm{d}z$ such that $\sqrt{|\det g_{\mu\nu}|} = 1$. Assume the orientation $\epsilon_{xyz} = +1$.

(a) Compute $\star\,\mathrm{d}x$, $\star\,\mathrm{d}y$, and $\star\,\mathrm{d}z$. [*Hint:* To find $\star\,\mathrm{d}x$, start with a 1-form $\underline{\omega} = \omega_\mu \mathrm{d}x^\mu$ with $\omega_x = 1$ and all other entries zero, and then insert it into the definition (2).]

(b) Compute $\star(\mathrm{d}x \wedge \mathrm{d}y)$, $\star(\mathrm{d}y \wedge \mathrm{d}z)$, and $\star(\mathrm{d}z \wedge \mathrm{d}x)$. [*Hint:* Again, it may be helpful to define a 2-form $\underline{\omega} = \frac{1}{2}\omega_{\mu\nu} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ with $\omega_{xy} = -\omega_{yx} = 1$ in combination with Eq. (2).]

(c) Compute the dual of the 3-form $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$.

Okay, this was not all for nothing. Let us use these results to derive some interesting vector calculus formulas! Start with the 1-form

$$\underline{A} = A_x \, \mathrm{d}x + A_y \, \mathrm{d}y + A_z \, \mathrm{d}z \, . \tag{3}$$

(d) Show that $\mathrm{d}\underline{A}$ corresponds to $\vec{\nabla} \times \vec{A}$.

(e) Show that $\mathrm{d}\mathrm{d}\underline{A} = 0$ corresponds to $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$.

(f) Show that $\mathrm{d} \star \underline{A}$ corresponds to $\vec{\nabla} \cdot \vec{A}$.

For this second part, let us explore how the Lorentz signature of Minkowski spacetime enters the Hodge dual. So, let us set $n = 4$ and assume that the metric is now given by $\underline{g} = -\mathrm{d}t \otimes \mathrm{d}t + \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}z \otimes \mathrm{d}z$ such that $\sqrt{|\det g_{\mu\nu}|} = 1$. Assume the orientation $\epsilon_{txyz} = +1$. Basically, we need to track the minus signs that arise from raising any index with a "t" in it in Eq. (2).

(g) Compute $\star\,\mathrm{d}t$, $\star\,\mathrm{d}x$, $\star\,\mathrm{d}y$, and $\star\,\mathrm{d}z$. [*Hint:* To find e.g. $\star\,\mathrm{d}t$, start with a 1-form $\underline{\omega} = \omega_\mu \mathrm{d}x^\mu$ with $\omega_t = 1$ and all other entries zero, and then insert it into the definition (2). Be mindful of the signs!]

(h) Compute $\star(\mathrm{d}t \wedge \mathrm{d}x)$, $\star(\mathrm{d}t \wedge \mathrm{d}y)$, $\star(\mathrm{d}t \wedge \mathrm{d}z)$.

(i) Compute $\star(\mathrm{d}x \wedge \mathrm{d}y)$, $\star(\mathrm{d}y \wedge \mathrm{d}z)$, $\star(\mathrm{d}z \wedge \mathrm{d}x)$.

These results will be helpful for the next exercise.

2 Another famous equation in disguise

Let us stay in four dimensions in the coordinates t, x, y, z. Consider the 2-form

$$\begin{aligned} \underline{F} = \quad & E_x\,\mathrm{d}x \wedge \mathrm{d}t \;+\; E_y\,\mathrm{d}y \wedge \mathrm{d}t \;+\; E_z\,\mathrm{d}z \wedge \mathrm{d}t \\ & + B_x\,\mathrm{d}y \wedge \mathrm{d}z \;+\; B_y\,\mathrm{d}z \wedge \mathrm{d}x \;+\; B_z\,\mathrm{d}x \wedge \mathrm{d}y\,, \end{aligned} \tag{4}$$

as well as the 1-form

$$\underline{j} = -\rho\,\mathrm{d}t + j_x\,\mathrm{d}x + j_y\,\mathrm{d}y + j_z\,\mathrm{d}z\,. \tag{5}$$

The functions E_x, E_y, E_z, B_x, B_y, B_z, ρ, j_x, j_y, and j_z are all functions of all coordinates t, x, y, z.

(a) Compute the Hodge dual $\star F$ and show that it is given by

$$\star\underline{F} = E_x\,\mathrm{d}y \wedge \mathrm{d}z + E_y\,\mathrm{d}z \wedge \mathrm{d}x + E_z\,\mathrm{d}x \wedge \mathrm{d}y - (B_x\,\mathrm{d}x + B_y\,\mathrm{d}y + \;B_z \mathrm{d}z) \wedge \mathrm{d}t\,. \tag{6}$$

Feel free to use the results from the previous exercise in this step.

(b) Compute the exterior derivative of the above expression, $\mathrm{d}\star\underline{F}$.

(c) Now, apply another Hodge dual and compute $\star\,\mathrm{d}\star\underline{F}$.

(d) Show that the equation $\star\,\mathrm{d}\star\underline{F} = \underline{j}$ amounts to the two inhomogeneous Maxwell equations $\vec{\nabla} \cdot \vec{E} = \rho$ and $\vec{\nabla} \times \vec{B} = \partial_t \vec{E} + \vec{j}$.

(e) Compute $\star j$ and show that the Maxwell equations imply $\mathrm{d}\star\underline{j} = 0$.

(f) Evaluate the equation $\mathrm{d}\star\underline{j} = 0$ and show that it corresponds to $\partial_t \rho + \vec{\nabla} \cdot \vec{j} = 0$.

(g) What is the name of this last equation?

3 What would you like to see in the last lectures?

For the last lectures, I want to address some more advanced topics that are fun to think about and may inspire you where you could go next after this course. Please select your two favorite topics :)

[ ] Differential forms in gauge theory
[ ] Curvature and torsion as differential forms
[ ] Topological phases and the Aharonov–Bohm effect
[ ] Black hole spacetimes