Issued: November 5, 2021
Due: 11am, November 12, 2021
Official website: http://spintwo.net/Courses/PHYS-581-Differential-Geometry-for-Physicists/

Please work on this problem set on your own; it should be possible to complete it with the lecture notes and no other external help. If you have questions you can email the instructor, Jens Boos (jboos@wm.edu), or make use of the office hours on Monday, 10am–11am, Small 235. *After* you have completed the assignment please feel free to discuss it with other students.

1 Quick questions

(a) Let $\underline{\omega}$ be a 1-form, $\underline{\lambda}$ be a 2-form, and $\underline{\phi}$ be a 0-form. Which statements are true?

[ ] $\underline{\omega} \wedge \underline{\omega} = 0$
[ ] $\underline{\lambda} \wedge \underline{\lambda} = 0$
[ ] $\underline{\phi} \wedge \underline{\lambda} = \underline{\lambda} \wedge \underline{\phi}$
[ ] $\underline{\omega} \wedge \underline{\lambda} = \underline{\lambda} \wedge \underline{\omega}$

(b) How many independent 2-forms are there in four dimensions? (*Hint:* think of coordinates t, x, y, z, and all possible combinations $\mathrm{d}x \wedge \mathrm{d}y$, and so on.)

[ ] 4
[ ] 6
[ ] 10
[ ] 16

(c) How many independent components does a p-form have in n dimensions?

[ ] $p!$
[ ] $n!$
[ ] $p!/(n-p)!$
[ ] $n!/[p!(n-p)!]$

(d) Which of these differential forms have vanishing exterior derivative?

[ ] $\mathrm{d}\varphi$
[ ] $\mathrm{d}x \wedge \mathrm{d}y$
[ ] $z\,\mathrm{d}x \wedge \mathrm{d}y$
[ ] 1

(e) In your own words, why does the exterior derivative of a p-form give rise to a tensorial object?

2 Differential forms

Let us work in a simple two-dimensional setting in the coordinates t and x. Consider the 1-form

$$\underline{\omega} = f\,\mathrm{d}t + h\,\mathrm{d}x\,, \tag{1}$$

where $f = f(t, x)$ and $h = h(t, x)$ are two arbitrary functions.

(a) Calculate $\underline{\lambda} = \mathrm{d}\underline{\omega}$ and show that

$$\underline{\lambda} = (\partial_t h - \partial_x f)\,\mathrm{d}t \wedge \mathrm{d}x\,. \tag{2}$$

(b) Show that $\mathrm{d}\underline{\lambda} = 0$.

(c) Find a special choice for the functions f and h such that $\underline{\lambda} = \mathrm{d}\underline{\omega} = 0$.

(d) If $\mathrm{d}\underline{\omega} = 0$, we call the 1-form $\underline{\omega}$ *closed*. As it turns out, this implies that there exists a 0-form $\underline{\alpha}$ such that $\underline{\omega} = \mathrm{d}\underline{\alpha}$. Given your choice for f and h above, find this 0-form $\underline{\alpha}$. (*Hint:* It is sufficient to demonstrate that the $\underline{\alpha}$ you find satisfies $\underline{\omega} = \mathrm{d}\underline{\alpha}$).

3 A famous equation in disguise

Let us now move to four dimensions in the coordinates t, x, y, z. Consider the 2-form

$$\begin{aligned}\underline{F} = \quad & E_x\,\mathrm{d}x \wedge \mathrm{d}t \;+\; E_y\,\mathrm{d}y \wedge \mathrm{d}t \;+\; E_z\,\mathrm{d}z \wedge \mathrm{d}t \\ & + B_x\,\mathrm{d}y \wedge \mathrm{d}z \;+\; B_y\,\mathrm{d}z \wedge \mathrm{d}x \;+\; B_z\,\mathrm{d}x \wedge \mathrm{d}y\,.\end{aligned} \tag{3}$$

The functions E_x, E_y, E_z, B_x, B_y, B_z are functions of all coordinates t, x, y, z. Now: the fun part!

(a) Compute the exterior derivative $\mathrm{d}\underline{F}$. You should obtain 12 non-vanishing terms.

(b) Split the terms in four groups by factoring out $\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y$, $\mathrm{d}t \wedge \mathrm{d}y \wedge \mathrm{d}z$, $\mathrm{d}t \wedge \mathrm{d}z \wedge \mathrm{d}x$, and $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$.

(c) Since we are in Minkowski spacetime we have the identity $E_x = E^x$, $B_x = B^x$, and so on. Rewrite these equations in terms of the vector components (E^x, E^y, E^z) and (B^x, B^y, B^z).

(d) Show that the $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ component of the equation $\mathrm{d}\underline{F} = 0$ corresponds to the equation $\nabla \cdot \boldsymbol{B} = 0$, that is, the divergence of the 3-vector $\boldsymbol{B}$.

(e) Show that the remaining three equations correspond to the equation $\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}$, where $\nabla \times \boldsymbol{E}$ denotes the curl of the 3-vector $\boldsymbol{E}$.

(f) What is the name of the equations we have just recovered?