Issued: October 29, 2021
Due: 11am, November 5, 2021
Official website: http://spintwo.net/Courses/PHYS-581-Differential-Geometry-for-Physicists/

Please work on this problem set on your own; it should be possible to complete it with the lecture notes and no other external help. If you have questions you can email the instructor, Jens Boos (jboos@wm.edu), or make use of the office hours on Monday, 10am–11am, Small 235. *After* you have completed the assignment please feel free to discuss it with other students.

1 Quick questions

(a) Which symmetries does a general curvature tensor $R_{\mu\nu\rho\sigma}$ satisfy?

[ ] $R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}$
[ ] $R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}$
[ ] $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$
[ ] $R_{\mu\nu\rho\sigma} = R_{\sigma\rho\nu\mu}$

(b) Which symmetries does the Riemann curvature tensor $\tilde{R}_{\mu\nu\rho\sigma}$ satisfy when there is no torsion?

[ ] $\tilde{R}_{\mu\nu\rho\sigma} = -\tilde{R}_{\nu\mu\rho\sigma}$
[ ] $\tilde{R}_{\mu\nu\rho\sigma} = -\tilde{R}_{\mu\nu\sigma\rho}$
[ ] $\tilde{R}_{\mu\nu\rho\sigma} = \tilde{R}_{\rho\sigma\mu\nu}$
[ ] $\tilde{R}_{\mu\nu\rho\sigma} = \tilde{R}_{\sigma\rho\nu\mu}$

(c) How many independent components does a general curvature tensor $R_{\mu\nu\rho\sigma}$ have in 4 dimensions?

[ ] 10
[ ] 20
[ ] 36
[ ] 4

(d) How many independent components does the Riemann tensor $\tilde{R}_{\mu\nu\rho\sigma}$ have in 4 dimensions?

[ ] 10
[ ] 20
[ ] 36
[ ] 4

(e) In your own words, why is there—in general—no unique connection on a differentiable manifold? And why does that mean that there is no unique curvature tensor?

2 Curvature of a two-dimensional metric

Consider the two-dimensional metric given by

$$\underline{g} = -f\mathrm{d}t \otimes \mathrm{d}t + \frac{1}{f}\mathrm{d}r \otimes \mathrm{d}r\,, \tag{1}$$

where $f = f(r)$. Our goal is to compute the scalar curvature $\tilde{R}$ for this metric.

(a) Write down the inverse metric coefficients $g^{\mu\nu}$.

(b) Recall that the Levi-Civita connection coefficients $\tilde{\Gamma}^{\mu}{}_{\nu\rho}$ (also frequently referred to as "Christoffel symbols") are given in terms of the metric via

$$\tilde{\Gamma}^{\mu}{}_{\nu\rho} = \frac{1}{2}g^{\mu\alpha}\left(\partial_\nu g_{\alpha\rho} + \partial_\rho g_{\alpha\nu} - \partial_\alpha g_{\nu\rho}\right)\,. \tag{2}$$

Using that $f = f(r)$, show that for the metric (1) the only non-vanishing coefficients are

$$\tilde{\Gamma}^{t}{}_{tr} = \frac{f'}{2f}\,, \quad \tilde{\Gamma}^{r}{}_{tt} = \frac{1}{2}ff'\,, \quad \tilde{\Gamma}^{r}{}_{rr} = -\frac{f'}{2f}\,, \tag{3}$$

where the prime denotes differentiation with respect to r.

(c) The Riemannian curvature tensor is

$$\tilde{R}_{\alpha\beta}{}^{\mu}{}_{\nu} = \partial_\alpha\tilde{\Gamma}^{\mu}{}_{\beta\nu} - \partial_\beta\tilde{\Gamma}^{\mu}{}_{\alpha\nu} + \tilde{\Gamma}^{\mu}{}_{\alpha\lambda}\tilde{\Gamma}^{\lambda}{}_{\beta\nu} - \tilde{\Gamma}^{\mu}{}_{\beta\lambda}\tilde{\Gamma}^{\lambda}{}_{\alpha\nu}\,. \tag{4}$$

In two dimensions, the curvature tensor has only one independent component. Show that

$$\tilde{R}_{tr}{}^{t}{}_{r} = -\frac{f''}{2f}\,, \tag{5}$$

where again the prime denotes differentiation with respect to r.

(d) What are the units of the curvature tensor?

(e) Now turn to the Ricci tensor defined via

$$\tilde{R}_{\mu\nu} = \tilde{R}_{\alpha\nu}{}^{\alpha}{}_{\mu}\,. \tag{6}$$

Show that

$$\tilde{R}_{tt} = \frac{f''f}{2} \quad \tilde{R}_{tr} = 0\,, \quad \tilde{R}_{rr} = -\frac{f''}{2f}\,. \tag{7}$$

(e) Finally we can calculate the Ricci scalar $\tilde{R} = g^{\mu\nu}\tilde{R}_{\mu\nu}$. Show that it is given by

$$\tilde{R} = -f''\,. \tag{8}$$

(f) What is the value of the curvature as $r \to \infty$, if f is a positive decreasing function of r?