Issued: October 22, 2021
Due: 11am, October 29, 2021
Official website: http://spintwo.net/Courses/PHYS-581-Differential-Geometry-for-Physicists/

Please work on this problem set on your own; it should be possible to complete it with the lecture notes and no other external help. If you have questions you can email the instructor, Jens Boos (jboos@wm.edu), or make use of the office hours on Monday, 10am–11am, Small 235. *After* you have completed the assignment please feel free to discuss it with other students.

1 Quick questions

(a) How does the connection $\Gamma^\mu{}_{\nu\rho}$ transform?

[ ] As a tensor.
[ ] Not as a tensor.
[ ] As a scalar.

(b) How do the torsion coefficients $T_{\mu\nu}{}^\lambda$ transform?

[ ] As a tensor.
[ ] Not as a tensor.
[ ] As a scalar.

(c) Which conditions are necessary to derive the Levi-Civita connection $\tilde{\Gamma}^\mu{}_{\nu\rho}$?

[ ] The existence of a metric.
[ ] Metric compatibility.
[ ] The vanishing of torsion.
[ ] The vanishing of curvature.

(d) The Levi-Civita connection is symmetric, $\tilde{\Gamma}^\mu{}_{\nu\rho} = \tilde{\Gamma}^\mu{}_{\rho\nu}$. How many independent components does this object have in n dimensions?

[ ] $n^2(n-1)/2$
[ ] $n^2(n+1)/2$
[ ] $n(n-1)/2 + n$
[ ] $n(n+1)/2 + n$

(e) Tensor algebra can be a bit confusing sometimes. Which of these equations are formally correct?

[ ] $v_\mu \omega_\nu = \omega_\nu v_\mu$
[ ] $X^{\alpha\beta}{}_\nu = v^\alpha \omega^\beta v_\nu$
[ ] $u_\mu v_\nu = u_\nu v_\mu$
[ ] $M_{\alpha\beta} v^\alpha v^\beta = M_{\alpha\beta} v^\beta v^\alpha$
[ ] $M_{\alpha\beta} u^\alpha v^\beta = M_{\alpha\beta} u^\beta v^\alpha$

2 Proof that the covariant derivative is really covariant

In the sixth lecture, on Oct 15, we proved that $\nabla_\rho v^\mu$ really transforms as a $\binom{1}{1}$ tensor under a coordinate transformation $x^\mu \to y^{\mu'}$. The algebra was a bit tedious, but it is one of the most important concepts we learn in this course. In this exercise, let us walk through the proof that the covariant derivative of a covector, $\nabla_\rho \omega_\mu$, transforms as a $\binom{0}{2}$ tensor under a coordinate transformation.

(a) Write down the expression for $\nabla_\rho \omega_\mu$ in terms of a partial derivative and the connection.

(b) Now consider a transformation to a new coordinate system $y^{\mu'}$ such that the partial derivative, the covector ω_μ, and the connection $\Gamma^\mu{}_{\nu\rho}$ transform as

$$\frac{\partial}{\partial x^\mu} = \frac{\partial y^{\mu'}}{\partial x^\mu}\frac{\partial}{\partial y^{\mu'}}\,, \tag{1}$$

$$\omega_\mu = \frac{\partial y^{\mu'}}{\partial x^\mu}\omega_{\mu'}\,, \tag{2}$$

$$\Gamma^\mu{}_{\nu\rho} = \frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial y^{\nu'}}{\partial x^\nu}\frac{\partial y^{\rho'}}{\partial x^\rho}\Gamma'^{\mu'}{}_{\nu'\rho'} + \frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial^2 y^{\mu'}}{\partial x^\nu \partial x^\rho}\,. \tag{3}$$

Also, for notational convenience, let us use the abbreviation

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}\,, \qquad \partial_{\mu'} \equiv \frac{\partial}{\partial y^{\mu'}}\,. \tag{4}$$

Using these formulas, show that the transformed expression can be written as

$$\nabla_\rho \omega_\mu = \frac{\partial y^{\rho'}}{\partial x^\rho}\frac{\partial y^{\mu'}}{\partial x^\mu}\left(\partial_{\rho'}\omega_{\mu'} - \Gamma'^{\alpha'}{}_{\rho'\mu'}\,\omega_{\alpha'}\right) + \frac{\partial y^{\rho'}}{\partial x^\rho}\frac{\partial^2 y^{\mu'}}{\partial y^{\rho'}\partial x^\mu}\omega_{\mu'} - \frac{\partial x^\alpha}{\partial y^{\alpha'}}\frac{\partial^2 y^{\alpha'}}{\partial x^\rho \partial x^\mu}\frac{\partial y^{\beta'}}{\partial x^\alpha}\omega_{\beta'}\,. \tag{5}$$

(c) Looking at (5), this transformation law is that of a tensor if and only if the last two terms cancel. Show that this is indeed the case.

(d) In your own words, what was the most challenging part of this computation for you? (Yes, this answer is graded and worth one point.)

You have now seen that both $\nabla_\rho v^\mu$ (in the lecture) and $\nabla_\rho \omega_\mu$ (right here on the assignment) transform as tensors. Since the covariant derivative of a general rank $\binom{p}{q}$ tensor is constructed successively by adding connection terms for each index of the differentiated object, our two examples should convince you that the covariant derivative of a rank $\binom{p}{q}$ tensor also transforms as a tensor.

3 Levi-Civita connection: ‘del’ in polar coordinates

Let us work with the Levi-Civita connection and show how it can be used to solve actual problems in physics. Consider the metric of flat space in polar coordinates,

$$\underline{g} = \mathrm{d}r \otimes \mathrm{d}r + r^2\, \mathrm{d}\varphi \otimes \mathrm{d}\varphi\,. \tag{6}$$

Our goal is to compute the covariant divergence of a vector field, $\nabla_\mu v^\mu$, as well as the covariant d’Alembert operator acting on a scalar field, $g^{\mu\nu}\nabla_\mu\nabla_\nu f$. Let us get there step by step.

(a) First, compute the coefficients of the inverse metric $g^{\mu\nu}$.

(b) The Levi-Civita connection is given by partial derivatives of the metric via

$$\tilde{\Gamma}^\mu{}_{\nu\rho} = \frac{1}{2} g^{\mu\alpha}\left(\partial_\nu g_{\alpha\rho} + \partial_\rho g_{\alpha\nu} - \partial_\alpha g_{\nu\rho}\right)\,. \tag{7}$$

Compute the six independent coefficients $\tilde{\Gamma}^r{}_{rr}$, $\tilde{\Gamma}^r{}_{r\varphi}$, $\tilde{\Gamma}^r{}_{\varphi\varphi}$, $\tilde{\Gamma}^\varphi{}_{rr}$, $\tilde{\Gamma}^\varphi{}_{r\varphi}$, $\tilde{\Gamma}^\varphi{}_{\varphi\varphi}$ and show that they are given by

$$\tilde{\Gamma}^r{}_{rr} = 0\,,\quad \tilde{\Gamma}^r{}_{r\varphi} = 0\,,\quad \tilde{\Gamma}^r{}_{\varphi\varphi} = -r\,,\quad \tilde{\Gamma}^\varphi{}_{rr} = 0\,,\quad \tilde{\Gamma}^\varphi{}_{r\varphi} = \frac{1}{r}\,,\quad \tilde{\Gamma}^\varphi{}_{\varphi\varphi} = 0\,. \tag{8}$$

(c) The covariant divergence of a vector field $\underline{v}$ is defined as

$$\nabla \cdot \underline{v} \equiv \delta^\mu_\nu\, \nabla_\mu v^\nu = \nabla_\mu v^\mu\,. \tag{9}$$

Show that it is equal to

$$\nabla_\mu v^\mu = \partial_\mu v^\mu + \tilde{\Gamma}^\mu{}_{\mu\alpha} v^\alpha\,. \tag{10}$$

(d) By inserting your results from (b), show the following relation:

$$\nabla_\mu v^\mu = \frac{\partial v^r}{\partial r} + \frac{\partial v^\varphi}{\partial \varphi} + \frac{1}{r} v^r \tag{11}$$

(e) Show that the second covariant derivative of a scalar function f is given by

$$\nabla_\mu \nabla_\nu f = \partial_\mu \partial_\nu f - \tilde{\Gamma}^\lambda{}_{\mu\nu} \partial_\lambda f\,. \tag{12}$$

(f) By multiplying this equation with $g^{\mu\nu}$ we can define the covariant d’Alembert operator,

$$\Box f \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu f = g^{\mu\nu}\left(\partial_\mu \partial_\nu f - \tilde{\Gamma}^\lambda{}_{\mu\nu} \partial_\lambda f\right)\,. \tag{13}$$

By inserting your results from (a) and (b), show the following relation:

$$\Box f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{1}{r}\frac{\partial f}{\partial r}\,. \tag{14}$$