Issued: October 1, 2021
Due: 11am, October 8, 2021
Official website: http://spintwo.net/Courses/PHYS-581-Differential-Geometry-for-Physicists/

Please work on this problem set on your own; it should be possible to complete it with the lecture notes and no other external help. If you have questions you can email the instructor, Jens Boos (jboos@wm.edu), or make use of the office hours on Monday, 10am–11am, Small 235. *After* you have completed the assignment please feel free to discuss it with other students.

1 Quick questions

(a) Select the equation(s) with mistakes in them.

[ ] $v_\mu = g_{\mu\nu}\, v^\mu$
[ ] $v_\mu = g_{\nu\mu}\, v^\mu$
[ ] $g_{\mu\nu}\, g^{\nu\rho} = \delta^\rho_\mu$
[ ] $M_\mu{}^\nu = g^{\mu\rho} M_{\rho\nu}$

(b) Which properties does a metric tensor $\underline{g}$ satisfy?

[ ] It is a symmetric $\binom{0}{2}$ tensor.
[ ] It is positive definite.
[ ] It is non-degenerate.
[ ] It is invertible.

(c) Select all expressions that are identical.

[ ] $T^\mu{}_{\nu\rho}\, \omega_\mu v^\nu v^\rho$
[ ] $T^\mu{}_{\rho\nu}\, \omega_\mu v^\nu v^\rho$
[ ] $T_{\mu\nu\rho}\, \omega^\mu v^\nu v^\rho$
[ ] $T_{\mu\nu\rho}\, \omega^\nu v^\mu v^\rho$

(d) Write down the norms of these tensors in terms of their components:

$$\begin{aligned}
\underline{v} &= v^\mu \partial_\mu\,, & |\underline{v}|^2 &= \\
\underline{\omega} &= \omega_\mu \mathrm{d}x^\mu\,, & |\underline{\omega}|^2 &= \\
\underline{T} &= T^{\mu\nu\rho}\, \partial_\mu \otimes \partial_\nu \otimes \partial_\rho\,, & |\underline{T}|^2 &= \\
\underline{R} &= R^{\mu\nu}{}_{\rho\sigma}\, \partial_\mu \otimes \partial_\nu \otimes \mathrm{d}x^\rho \otimes \mathrm{d}x^\sigma\,, & |\underline{R}|^2 &=
\end{aligned}$$

(e) Insert multiplications with the metric and the inverse metric to obtain correct equations:

$$\begin{aligned}
v_\mu &= \qquad\qquad v^\rho\,, \\
T_{\mu\nu} &= \qquad\qquad T^{\rho\sigma}\,, \\
T^{\alpha\beta}{}_{\mu\nu} &= \qquad\qquad T_\lambda{}^\beta{}_{\mu\nu}\,, \\
v_\mu v_\nu &= \qquad\qquad v^\alpha v^\beta\,.
\end{aligned}$$

2 The metric of a black hole

The metric in the presence of a non-rotating black hole of mass m, expressed in spherical coordinates (t, r, θ, φ) is given by (we use units wherein $G = c = 1$)

$$\begin{aligned} \underline{g} &= g_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \\ &= -\left(1 - \frac{2m}{r}\right) \mathrm{d}t \otimes \mathrm{d}t + \frac{1}{1 - \frac{2m}{r}} \mathrm{d}r \otimes \mathrm{d}r + r^2 \mathrm{d}\theta \otimes \mathrm{d}\theta + r^2 \sin^2\theta \mathrm{d}\varphi \otimes \mathrm{d}\varphi\,. \end{aligned} \tag{1}$$

(a) Read off the components of $\underline{g}$ in the coordinate basis

$$\begin{aligned} g_{tt} &= \quad, & g_{tr} &= \quad, & g_{t\theta} &= \quad, & g_{t\varphi} &= \quad, \\ g_{rt} &= \quad, & g_{rr} &= \quad, & g_{r\theta} &= \quad, & g_{r\varphi} &= \quad, \\ g_{\theta t} &= \quad, & g_{\theta r} &= \quad, & g_{\theta\theta} &= \quad, & g_{\theta\varphi} &= \quad, \\ g_{\varphi t} &= \quad, & g_{\varphi r} &= \quad, & g_{\varphi\theta} &= \quad, & g_{\varphi\varphi} &= \quad. \end{aligned}$$

(b) What form does the metric take as $r \to \infty$? *Optional:* Do you recognize this form of the metric from somewhere? Metrics that have this property are said to be *asymptotically flat.*

(c) What are the components of the inverse metric $\underline{g}^{-1} = g^{\mu\nu}\partial_\mu \otimes \partial_\nu$?

$$\begin{aligned} g^{tt} &= \quad, & g^{tr} &= \quad, & g^{t\theta} &= \quad, & g^{t\varphi} &= \quad, \\ g^{rt} &= \quad, & g^{rr} &= \quad, & g^{r\theta} &= \quad, & g^{r\varphi} &= \quad, \\ g^{\theta t} &= \quad, & g^{\theta r} &= \quad, & g^{\theta\theta} &= \quad, & g^{\theta\varphi} &= \quad, \\ g^{\varphi t} &= \quad, & g^{\varphi r} &= \quad, & g^{\varphi\theta} &= \quad, & g^{\varphi\varphi} &= \quad. \end{aligned}$$

(d) Consider now the vector field $\underline{\xi} = \xi^\mu \partial_\mu = \partial_t$. What are its components ξ^μ? Compute its norm $|\underline{\xi}|^2 = g_{\mu\nu}\,\xi^\mu \xi^\nu$. What happens to the norm of this vector field at $r = 2m$?

(e) Find a new radial coordinate r_* that satisfies $\frac{\mathrm{d}r}{\mathrm{d}r_*} = 1 - \frac{2m}{r}$. This coordinate r_* is often called the "tortoise coordinate" in the literature. How does it behave at $r = 2m$?

(f) Using the result of (e), that is, $r_* = r + 2m \log\left(\frac{r}{2m} - 1\right)$, plot the dimensionless quantity $r_*/(2m)$ as a function of $r/(2m)$, where $r/(2m) \in (1, \infty)$. What happens as $r/(2m) \to \infty$?

(g) Perform a coordinate transformation from $x^\mu = (t, r, \theta, \varphi)$ to $y^{\mu'} = (t, r_*, \theta, \varphi)$ and show that the resulting metric is given by

$$\begin{aligned} \underline{g} &= g_{\mu'\nu'}\, \mathrm{d}y^{\mu'} \otimes \mathrm{d}y^{\nu'} \\ &= \left[1 - \frac{2m}{r(r_*)}\right] (-\mathrm{d}t \otimes \mathrm{d}t + \mathrm{d}r_* \otimes \mathrm{d}r_*) + r^2(r_*)\mathrm{d}\theta \otimes \mathrm{d}\theta + r^2(r_*) \sin^2\theta \mathrm{d}\varphi \otimes \mathrm{d}\varphi\,. \end{aligned} \tag{2}$$

Hint: You do not need to express r in terms of r_* explicitly, which in general is complicated. Instead, based on your plot in (f), argue that the inverse function $r(r_*)$ always exists.