Issued: September 24, 2021
Due: 11am, October 1, 2021
Official website: http://spintwo.net/Courses/PHYS-581-Differential-Geometry-for-Physicists/

Please work on this problem set on your own; it should be possible to complete it with the lecture notes and no other external help. If you have questions you can email the instructor, Jens Boos (jboos@wm.edu), or make use of the office hours on Monday, 10am–11am, Small 235 (*note: room change*). *After* you have completed the assignment please feel free to discuss it with other students.

1 Quick questions

(a) Which equations have mistakes in them?

[ ] $v^\mu = e^\mu{}_i \, v^i$
[ ] $v^\mu = e^\mu_i \, v^i$
[ ] $T_{ijk} = e^\mu{}_i \, e^\nu{}_j \, e^\rho{}_k \, T_{\mu\nu\rho}$
[ ] $M^i{}_j = e_\mu{}^i \, e^\nu{}_j \, M^\mu_\nu$

(b) Which of the following components belong to the tensor $\underline{T} = x\partial_x \otimes \mathrm{d}y - 2\partial_y \otimes \mathrm{d}z$?

[ ] $T^x{}_y = x$
[ ] $T^y{}_x = x$
[ ] $T^z{}_z = 0$
[ ] $T^y{}_z = -2$

(c) Select all expressions that are identical.

[ ] $T^i{}_j \, \hat{e}_i \otimes \hat{\vartheta}^j$
[ ] $T^\mu{}_j \, \partial_\mu \otimes \hat{\vartheta}^j$
[ ] $T^i{}_\nu \, \hat{e}_i \otimes \mathrm{d}x^\nu$
[ ] $T^\mu{}_\nu \, \partial_\mu \otimes \mathrm{d}x^\nu$

(d) A $\binom{2}{0}$ tensor $\underline{T}$ is antisymmetric if its components satisfy $T^{\mu\nu} = -T^{\nu\mu}$. Which of these tensors are antisymmetric?

[ ] $\underline{T} = x\partial_x \otimes \partial_y - y\partial_y \otimes \partial_x$
[ ] $\underline{T} = x\partial_x \otimes \partial_y$
[ ] $\underline{T} = 0$
[ ] $\underline{T} = x(\partial_z \otimes \partial_y - \partial_y \otimes \partial_z)$

(e) How many independent components does an antisymmetric $\binom{2}{0}$ tensor have in n dimensions?

[ ] 3
[ ] n^2
[ ] $n(n-1)$
[ ] $n(n-1)/2$

2 Frames and coordinates

In the last assignment we introduced the tensor $\underline{\eta}$, which in the coordinates $x^\mu = (r, \theta, \varphi)$ is

$$\underline{\eta} = \eta_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \mathrm{d}r \otimes \mathrm{d}r + r^2 \mathrm{d}\theta \otimes \mathrm{d}\theta + r^2 \sin^2\theta \mathrm{d}\varphi \otimes \mathrm{d}\varphi \,. \tag{1}$$

(a) Consider now the abstract cobasis $\hat{\vartheta}^j$ (with $j = 1, 2, 3$) given by

$$\hat{\vartheta}^1 = \mathrm{d}r\,, \quad \hat{\vartheta}^2 = r\mathrm{d}\theta\,, \quad \hat{\vartheta}^3 = r\sin\theta\mathrm{d}\varphi\,. \tag{2}$$

Recall that we defined the object ${e_\mu}^j$ via $\hat{\vartheta}^j = {e_\mu}^j \mathrm{d}x^\mu$. Read off the coefficients ${e_\mu}^j$.

$$\begin{aligned} {e_r}^1 &= \qquad, & {e_r}^2 &= \qquad, & {e_r}^3 &= \qquad, \\ {e_\theta}^1 &= \qquad, & {e_\theta}^2 &= \qquad, & {e_\theta}^3 &= \qquad, \\ {e_\varphi}^1 &= \qquad, & {e_\varphi}^2 &= \qquad, & {e_\varphi}^3 &= \qquad. \end{aligned}$$

(b) Show by direct substitution that

$$\underline{\eta} = \hat{\vartheta}^1 \otimes \hat{\vartheta}^1 + \hat{\vartheta}^2 \otimes \hat{\vartheta}^2 + \hat{\vartheta}^3 \otimes \hat{\vartheta}^3\,. \tag{3}$$

(c) Read off the components of $\underline{\eta}$ in this abstract basis:

$$\begin{aligned} \eta_{11} &= \quad, & \eta_{12} &= \quad, & \eta_{13} &= \quad, \\ \eta_{21} &= \quad, & \eta_{22} &= \quad, & \eta_{23} &= \quad, \\ \eta_{31} &= \quad, & \eta_{32} &= \quad, & \eta_{33} &= \quad. \end{aligned}$$

(d) The abstract basis and cobasis are dual to one another such that $\hat{e}_i \lrcorner \hat{\vartheta}^j = \delta^j_i$. Find the expressions for $\hat{e}_i$ expanded in terms of the coordinate basis ∂_μ and verify that they are dual to one another. Example: $\hat{e}_1 = \partial_r$ such that $\hat{e}_1 \lrcorner \hat{\vartheta}^1 = \partial_r \lrcorner \mathrm{d}r = \delta^r_r = 1$.

(e) The abstract basis $\hat{e}_i$ is related to the coordinate basis ∂_μ via $\hat{e}_i = {e^\mu}_i \partial_\mu$. Using the results from (d), read off the "vielbein" coefficients ${e^\mu}_i$.

$$\begin{aligned} {e^r}_1 &= \qquad, & {e^r}_2 &= \qquad, & {e^r}_3 &= \qquad, \\ {e^\theta}_1 &= \qquad, & {e^\theta}_2 &= \qquad, & {e^\theta}_3 &= \qquad, \\ {e^\varphi}_1 &= \qquad, & {e^\varphi}_2 &= \qquad, & {e^\varphi}_3 &= \qquad. \end{aligned}$$

(f) Finally, verify that the coefficients ${e^\mu}_i$ and ${e_\nu}^j$ really satisfy ${e^\mu}_i {e_\mu}^j = \delta^j_i$ as well as ${e^\mu}_i {e_\nu}^i = \delta^\mu_\nu$.