Issued: September 17, 2021
Due: 11am, September 24, 2021
Official website: http://spintwo.net/Courses/PHYS-581-Differential-Geometry-for-Physicists/

Please work on this problem set on your own; it should be possible to complete it with the lecture notes and no other external help. If you have questions you can email the instructor, Jens Boos (jboos@wm.edu), or make use of the office hours on Monday, 10am–11am, Small 340. *After* you have completed the assignment please feel free to discuss it with other students.

1 Quick questions

(a) Which equations have mistakes in them?

[ ] $\underline{v} = y\partial_x - x\partial_y$
[ ] $\underline{M} = \partial_y \otimes \mathrm{d}y - 27z^2\partial_z \otimes \mathrm{d}x$
[ ] $\underline{T} = T^\alpha{}_{\beta\gamma}\mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta \otimes \mathrm{d}x^\gamma$
[ ] $\underline{T} = T^\alpha{}_{\beta\gamma}\partial_\alpha \otimes \mathrm{d}x^\beta \otimes \mathrm{d}x^\gamma$

(b) What rank $\binom{p}{q}$ do the following tensor components have?

[ ] $M^i{}_j$
[ ] $T^{ijk}{}_{lmnpqr}$
[ ] δ^i_j
[ ] η_{ij}

(c) Select all expressions that are identical.

[ ] $T^i{}_j\,\omega_i\,v^j$
[ ] $T^\alpha{}_\beta\,\omega_\alpha\,v^\beta$
[ ] $T^i{}_k\,\omega_i\,v^k$
[ ] $T^k{}_i\,v^i\,\omega_k$

(d) A $\binom{2}{0}$ tensor $\underline{T}$ is symmetric if its components satisfy $T^{\mu\nu} = T^{\nu\mu}$. Which of these tensors are symmetric?

[ ] $\underline{T} = x\partial_x \otimes \partial_y - \partial_y \otimes \partial_x$
[ ] $\underline{T} = x\partial_x \otimes \partial_y$
[ ] $\underline{T} = 0$
[ ] $\underline{T} = x(\partial_x \otimes \partial_y + \partial_y \otimes \partial_x)$

(e) How many independent components does a symmetric tensor have in n dimensions?

[ ] 2
[ ] n^2
[ ] $n(n+1)$
[ ] $n(n+1)/2$

2 Coordinate transformations

(a) Consider the $\binom{0}{2}$ tensor $\underline{\eta}$ expressed in the coordinates $\{x, y, z\}$:

$$\underline{\eta} = \eta_{\mu\nu}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}z \otimes \mathrm{d}z\,. \tag{1}$$

Read off all of its components in the $\{x, y, z\}$ coordinates:

$$\begin{aligned} \eta_{xx} &= \quad, & \eta_{xy} &= \quad, & \eta_{xz} &= \quad, \\ \eta_{yx} &= \quad, & \eta_{yy} &= \quad, & \eta_{yz} &= \quad, \\ \eta_{zx} &= \quad, & \eta_{zy} &= \quad, & \eta_{zz} &= \quad. \end{aligned}$$

(b) Transform the above tensor to the new coordinates $\{r, \theta, \varphi\}$ given by

$$x = r\sin\theta\cos\varphi\,, \quad y = r\sin\theta\sin\varphi\,, \quad z = r\cos\theta\,, \tag{2}$$

and show that one obtains

$$\underline{\eta} = \mathrm{d}r \otimes \mathrm{d}r + r^2\,\mathrm{d}\theta \otimes \mathrm{d}\theta + r^2\sin^2\theta\,\mathrm{d}\varphi \otimes \mathrm{d}\varphi\,. \tag{3}$$

Hint: Start with Eq. (1) and use relations of the form $\mathrm{d}x = \frac{\partial x}{\partial r}\mathrm{d}r + \frac{\partial x}{\partial \theta}\mathrm{d}\theta + \frac{\partial x}{\partial \varphi}\mathrm{d}\varphi$, and similar for y and z, making use of the transformations (2).

(c) Read off the components of $\underline{\eta}$ in the $\{r, \theta, \varphi\}$ coordinates:

$$\begin{aligned} \eta_{rr} &= \quad, & \eta_{r\theta} &= \quad, & \eta_{r\varphi} &= \quad, \\ \eta_{\theta r} &= \quad, & \eta_{\theta\theta} &= \quad, & \eta_{\theta\varphi} &= \quad, \\ \eta_{\varphi r} &= \quad, & \eta_{\varphi\theta} &= \quad, & \eta_{\varphi\varphi} &= \quad. \end{aligned}$$

(d) *Optional:* Do you know what this tensor $\underline{\eta}$ is called?

3 More coordinate transformations

Consider the vector field $\underline{v}$ which in Cartesian coordinates $\{x, y\}$ is given by $\underline{v} = x\partial_y - y\partial_x$. We use here the shorthand notation $\partial_x = \frac{\partial}{\partial x}$, and so on.

(a) What are the components of this vector field?

$$v^x = \qquad, \qquad v^y = \qquad.$$

(b) Visualize this vector field in the xy-plane.

(c) Construct a covector field $\underline{\omega}$ that satisfies $\underline{\omega}(\underline{v}) = x^2 + y^2$?

(d) Consider now the polar coordinates $\{\rho, \varphi\}$ where $x = \rho\cos\varphi$ and $y = \rho\sin\varphi$, and express the vector field $\underline{v}$ in these coordinates. *Hint:* Use $\partial_x = \frac{\partial \rho}{\partial x}\partial_\rho + \frac{\partial \varphi}{\partial x}\partial_\varphi$ and $\partial_y = \frac{\partial \rho}{\partial y}\partial_\rho + \frac{\partial \varphi}{\partial y}\partial_\varphi$.

(e) In your diagram of task (b), visualize the local direction of the basis vector fields ∂_ρ and ∂_φ. (We have not yet discussed the metric, so feel free to draw all of these vectors with unit length.)