Issued: December 3, 2021
Due: 11am, December 10, 2021
Official website: http://spintwo.net/Courses/PHYS-581-Differential-Geometry-for-Physicists/

Please work on this problem set on your own; it should be possible to complete it with the lecture notes and no other external help. If you have questions you can email the instructor, Jens Boos (jboos@wm.edu), or make use of the office hours on Monday, 10am–11am, Small 235. *After* you have completed the assignment please feel free to discuss it with other students.

1 Embedded sphere

Consider the following embedding function $\vec{\Psi}$ which maps two coordinates $\{\theta, \varphi\}$ to the surface of the 2-sphere in $\mathbb{R}^3$ (in Euclidean coordinates, and r is a constant of dimension length):

$$\vec{\Psi}(\theta, \varphi) = r(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\,. \tag{1}$$

(a) Compute the tangential vectors $\partial_\theta \equiv \frac{\partial\vec{\Psi}}{\partial\theta}$ and $\partial_\varphi \equiv \frac{\partial\vec{\Psi}}{\partial\varphi}$.

(b) Compute the components of the metric $g_{\mu\nu} \equiv \langle \frac{\partial\vec{\Psi}}{\partial x^\mu}, \frac{\partial\vec{\Psi}}{\partial x^\nu} \rangle$ and show that

$$g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = r^2(\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta\mathrm{d}\varphi \otimes \mathrm{d}\varphi)\,. \tag{2}$$

(c) Argue that this metric is that of a sphere of radius r.

2 Embedded cylinder

Consider the following embedding function $\vec{\Psi}$ which maps two coordinates $\{\varphi, z\}$ to the surface of a cylinder in $\mathbb{R}^3$ (in Euclidean coordinates, and r is again a constant of dimension length):

$$\vec{\Psi}(\varphi, z) = (r\cos\varphi, r\sin\varphi, z)\,. \tag{3}$$

(a) Compute the tangential vectors $\partial_\varphi \equiv \frac{\partial\vec{\Psi}}{\partial\varphi}$ and $\partial_z \equiv \frac{\partial\vec{\Psi}}{\partial z}$.

(b) Compute the components of the metric $g_{\mu\nu} \equiv \langle \frac{\partial\vec{\Psi}}{\partial x^\mu}, \frac{\partial\vec{\Psi}}{\partial x^\nu} \rangle$ and show that

$$g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = r^2\mathrm{d}\varphi \otimes \mathrm{d}\varphi + \mathrm{d}z \otimes \mathrm{d}z\,. \tag{4}$$

(c) Argue that this metric is that of a cylinder of radius r.

(d) Prove that the scalar curvature of this metric is zero.