Issued: September 10, 2021
Due: 11am, September 17, 2021
Official website: http://spintwo.net/Courses/PHYS-581-Differential-Geometry-for-Physicists/

Please work on this problem set on your own; it should be possible to complete it with the lecture notes and no other external help. If you have questions you can email the instructor, Jens Boos (jboos@wm.edu), or make use of the office hours on Monday, 10am–11am, Small 340. *After* you have completed the assignment please feel free to discuss it with other students.

1 Quick questions

(a) Select all items that are *not* a vector.

[ ] velocity
[ ] temperature
[ ] acceleration
[ ] energy

(b) Which equations have mistakes in them?

[ ] $\underline{v} = v^i \hat{e}^i$
[ ] $\underline{v} = v^i \hat{e}_i$
[ ] $\underline{M} = M^{ij} \hat{\vartheta}_i \otimes \hat{\vartheta}_k$
[ ] $\underline{T} = T^{ijk} \hat{\vartheta}_i \otimes \hat{\vartheta}_j$

(c) Which properties does a vector space have to satisfy?

[ ] Adding two elements of the vector space yields another element of the vector space.
[ ] Multiplying an element of the vector space with an element of the field ($\mathbb{R}$ or $\mathbb{C}$) yields another element of the vector space.
[ ] You can divide elements by one another.
[ ] There *has to be* a product that maps two elements of the vector space into the field ($\mathbb{R}$ or $\mathbb{C}$).

(d) In the notation of quantum mechanics, which of the following objects is a rank $\binom{0}{2}$ tensor?

[ ] $\langle\phi| \otimes \langle\psi|$
[ ] $|\chi\rangle \otimes \langle\phi|$
[ ] $|\chi\rangle \otimes |\lambda\rangle$
[ ] $\langle\phi|\chi\rangle$

(e) How many independent components does a rank $\binom{2}{3}$ tensor have in n dimensions?

[ ] $5n$
[ ] $n^2 + n^3$
[ ] $(5n)!$
[ ] n^5

2 Tensor algebra

Let $\underline{v}$ be a vector, $\underline{\omega}$ be a covector, $\underline{M}$ be a $\binom{0}{2}$ tensor, $\underline{F}$ be a $\binom{2}{0}$ tensor, and $\underline{T}$ be a $\binom{2}{2}$ tensor. The basis is called $\hat{e}_i$ and the cobasis is called $\hat{\vartheta}^i$, and we work in n dimensions.

(a) Expand $\underline{v}$, $\underline{\omega}$, $\underline{M}$, $\underline{F}$, and $\underline{T}$ in this basis.

(b) Why is $\underline{M} + \underline{F}$ *not* a tensor?

(c) In the lecture we learned how to use tensor contraction to create scalar quantities. Write down 3 such scalar quantities of your choice involving the symbols $\underline{v}$, $\underline{\omega}$, $\underline{M}$, $\underline{F}$, and $\underline{T}$ and give the resulting expression in components. (Example: $\underline{M}(\underline{v}, \underline{v}) = M_{ij} v^i v^j$.)