when	weekly, 2pm, 4-285 CCIS
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URL	http://www.spintwo.net/Courses/Gauge-Theory-Student-Meetings/

5 A mini-introduction to Lie groups

The notion of groups is ubiquitous in physics, and in gauge theory they play a central role as well (in particular, as we will see, the notion of *Lie groups*). Recall from the last meeting the Noether current,

$$\partial_\mu \left[\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^A)} \left(\delta\phi^A - \boldsymbol{L}_{\delta x}\phi^A \right) + \mathscr{L}\delta x^\mu \right] = 0 \quad \text{on-shell} \,. \tag{1}$$

As we have seen, it is conserved due to the existence of a continuous symmetry. Moreover, it is important to stress that for each functionally different symmetry one obtains one conserved physical current: in case of the spacetime translations, of which there are four, we showed the conservation of energy and momentum (four quantities). From the U(1) symmetry of the charged scalar field, for example, we only got one conserved current (which is fine, since a U(1) transformation is parametrized by one continuous parameter).

As another example, consider this Lagrangian for three complex scalar fields ϕ, ψ, and χ:

$$\mathscr{L} = -(\partial_\mu\phi)(\partial^\mu\phi^*) - (\partial_\mu\psi)(\partial^\mu\psi^*) - (\partial_\mu\chi)(\partial^\mu\chi^*) - V(|\phi|^2, |\psi|^2, |\chi|^2) \,, \tag{2}$$

which is clearly invariant under

$$\begin{aligned} \phi'(x) &= e^{i\alpha}\phi(x) \,, \quad \phi^{*\prime}(x) = e^{-i\alpha}\phi^*(x) \,, \quad \alpha \in \mathbb{R} = \text{const} \,, \\ \psi'(x) &= e^{i\beta}\psi(x) \,, \quad \psi^{*\prime}(x) = e^{-i\beta}\psi^*(x) \,, \quad \beta \in \mathbb{R} = \text{const} \,, \\ \chi'(x) &= e^{i\gamma}\chi(x) \,, \quad \chi^{*\prime}(x) = e^{-i\gamma}\chi^*(x) \,, \quad \gamma \in \mathbb{R} = \text{const} \,. \end{aligned} \tag{3}$$

It is quite obvious that now there are three conserved currents, isn't it? They read

$$j_1^\mu = i\left(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi\right) \,, \tag{4}$$

$$j_2^\mu = i\left(\psi\partial^\mu\psi^* - \psi^*\partial^\mu\psi\right) \,, \tag{5}$$

$$j_3^\mu = i\left(\chi\partial^\mu\chi^* - \chi^*\partial^\mu\chi\right) \,. \tag{6}$$

So let us remember that *each functionally independent symmetry property* gives rise to one conserved current. This corresponds exactly to us factoring out the parameters ϵ, ϵ^μ, or $\omega^{\mu\nu}$ out of the Noether current j^μ in the examples of the last meeting. We will make this more precise as we move along.

At any rate, we see that symmetry transformations that can be described in terms of various *continuous parameters* are important in connection with Noether's theorem. Mathematically, these can be described in terms of *Lie groups*. Before going there, let us first understand what a *group* is.

Formally, a group $\mathcal{G}$ is a set of elements that satisfy the following relations:

(i) *Closure.* For any $g_1, g_2 \in \mathcal{G}$ one has $g_1 \circ g_2 \in \mathcal{G}$
(ii) *Associativity.* For any $g_1, g_2, g_3 \in \mathcal{G}$ one has $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$.
(iii) *Neutral element.* There is one $e \in \mathcal{G}$ such that for any $g \in \mathcal{G}$ one has $g \circ e = e \circ g = g$.
(iv) *Inverse.* Every element $g \in \mathcal{G}$ has an inverse $g^{-1} \in \mathcal{G}$ such that $g \circ g^{-1} = g^{-1} \circ g = e$.

Moreover, in an *Abelian group* all group elements commute, $g_i \circ g_j = g_j \circ g_i$ for all $g_i, g_j \in \mathcal{G}$. Conversely, in a *non-Abelian group*, there exists at least two elements that do *not* commute.

Note that we did not specify the number of elements in $\mathcal{G}$. It can be a discrete set of, say, five elements, but it can also be an infinite set of elements that are labelled by some continuous parameter(s). Or it can be a combination of continuous and discrete elements.

5.1 Group representations

Group elements $g \in \mathcal{G}$ are abstract mathematical objects. In physics, however, it is quite useful to imagine them as operators that act on some physical object (say, a matter wave function, or a vector). In particular, many group elements (and therefore many groups) can be described as matrices (and therefore a group can be described as a set of matrices). Mapping group elements to a matrix (or some other linear operator) is called a *representation* of that group element, and we should remember that one group $\mathcal{G}$ can have many possible representations. That being said, for many practical purposes it is admissible to think of a group as a set of matrices, which is what we will mostly do in what follows.

5.2 Examples of well-known groups

Let us list a few well-known groups and describe what they are:

- Unitary group U(N): the set of unitary $N \times N$ matrices ($M^\dagger = M^{-1}$)
- Unitary group U(1): the set of unitary "1×1 matrices" (better known as complex numbers) that can be written as $\exp(i\alpha)$ with $\alpha \in \mathbb{R}$. One can think of it as the set of complex phases.
- Special unitary group SU(N): the set of unitary $N \times N$ matrices that have determinant $+1$
- Orthogonal group O(N): the set of real, orthogonal matrices ($M^\mathrm{T} = M^{-1}$). The group O(3) corresponds to rotations (determinant $+1$) or reflections (determinant -1) in three dimensions.
- Special orthogonal group SO(N): the set of real, orthogonal matrices ($M^\mathrm{T} = M^{-1}$) with determinant $+1$.
- Lorentz group SO(1,3): the same as O(N), but defined on $(3+1)$-dimensional Minkowski space
- General linear group GL(N, $\mathbb{R}$): set of all real $N \times N$ matrices

As it turns out, all of these groups are Lie groups. Moreover, it can be shown that a product of groups is also a group. For example, the Lagrangian with the three complex scalar fields ϕ, ψ, and χ has the symmetry group U(1)×U(1)×U(1).

5.3 Lie groups

As we have said before, Lie groups have the property that its group elements can be labelled by a continuous parameter.[1] Mathematically speaking, Lie groups admit a differentiable structure and can be thought of as a manifold. Let us denote these "continuous labels" by the abstract vector α^I, where $I = 1, 2, \dots$ runs over as many components as we need. For the group U(1), α^I is just one number, but for the rotation group SO(3), say, α^I is the collection of all three rotation angles.

A group element (labelled by α) can now be written as

$$g(\alpha) = \exp\left(\sum_{I=1}^{N} \alpha^I \hat{t}^I\right) , \tag{7}$$

where the set $\{\hat{t}^I\}$ are the N so-called *generators* of the Lie group, the exponential is to be thought of as a formal expression that has to be evaluated using the exponential power series. We should think of these generators as operators. For an Abelian group these commute, whereas for a non-Abelian group they don't:

$$\text{Abelian}: \quad [\hat{t}^i, \hat{t}^j] = 0 , \qquad \text{non-Abelian}: \quad [\hat{t}^i, \hat{t}^j] = \sum_{k=1}^{N} f^{ijk} \hat{t}^k . \tag{8}$$

The constants f^{ijk} are called the structure constants of the group. Once we have a matrix representation of a group, we usually also have a matrix representation of the generators. Then, we can calculate the symbols f^{ijk} explicitly (they are just real or complex numbers). Let us consider some examples:[2]

5.3.1 Example: translations

In order to understand generators better, let us consider the Abelian group of translations (because it is Abelian, we can think of one-dimensional translations or 298-dimensional translations, it doesn't matter). In what follows, we can prove that translations are generated by the partial derivatives: this can be seen in close analogy to quantum mechanics, wherein the real-space representation of the momentum operator is given by the partial derivative.

[1] We do not aim to give a precise mathematical introduction into Lie groups in these notes, rather we would like to build some physical intuition. Please contact me if you find a mathematical blunder.

[2] More examples in [1].

Let us construct a translation acting on a function $f(x)$ that shifts the function to $f(x+\alpha)$:

$$\begin{aligned}\exp\left(\alpha\frac{\partial}{\partial x}\right)f(x) &= \left[\sum_{k=0}^{\infty}\frac{\alpha^k}{k!}\frac{\partial^k}{\partial x^k}\right]\left[\sum_{n=0}^{\infty}\frac{f_n}{n!}x^n\right] = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{\alpha^k f_n}{k!n!}\frac{\partial^k x^n}{\partial x^k} = \sum_{k=0}^{n}\sum_{n=0}^{\infty}\frac{\alpha^k f_n}{k!n!}\frac{n!}{(n-k)!}x^{n-k} \\ &= \sum_{n=0}^{\infty}\frac{f_n}{n!}\sum_{k=0}^{n}\binom{n}{k}\alpha^k x^{n-k} = \sum_{n=0}^{\infty}\frac{f_n}{n!}(x+\alpha)^n = f(x+\alpha)\end{aligned} \tag{9}$$

In the first step we inserted the expansion of the exponential as well as that of the function $f(x)$, and then made use of

$$\frac{\partial^n x^m}{\partial x^n} = \begin{cases}\frac{m!}{(m-n)!}x^{m-n} & \text{for } n > m \\ 0 & \text{else}\end{cases}, \tag{10}$$

and in the last equality we used the binomial theorem. Similarly, an n-dimensional translation in the direction $\vec{\alpha} = (\alpha^1, \alpha^2, \ldots, \alpha^n)$ is then given by

$$f(\vec{x}+\vec{\alpha}) = \exp\left(\vec{\alpha}\cdot\vec{\nabla}\right)f(\vec{x})\,. \tag{11}$$

It is also see from this that translations form an Abelian group, since they are generated by partial derivatives (which commute):

$$[\partial_i, \partial_j] = 0 \quad \leftrightarrow \quad f^{ijk} = 0\,. \tag{12}$$

In fact, it is that property that allows us to write

$$\exp\left(\vec{a}\cdot\vec{\nabla}\right)f(\vec{x}) = \left[\prod_{i=1}^{n}\exp\left(a^i\partial_i\right)\right]f(\vec{x})\,, \tag{13}$$

where there is no summation inside the exponential.

5.3.2 Example: 2D rotations

The two-dimensional rotations are an Abelian group as well, and it has only one generator: an antisymmetric 2×2 matrix. Let us prove it:

$$\begin{aligned}\exp\left[\alpha\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right] &= \sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}^n = \sum_{n=0}^{\infty}(-1)^n\frac{\alpha^{2n}}{(2n)!}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \sum_{n=0}^{\infty}(-1)^n\frac{\alpha^{2n+1}}{(2n+1)!}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \\ &= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\sum_{n=0}^{\infty}(-1)^n\frac{\alpha^{2n}}{(2n)!} + \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\sum_{n=0}^{\infty}(-1)^n\frac{\alpha^{2n+1}}{(2n+1)!} = \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\,.\end{aligned} \tag{14}$$

In the second equality we split the sum into its even and odd summands and used the fact that the generator $\hat{t}$ squares to itself, up to a sign, whereas all even powers of it are proportional to the identity matrix (up to a sign). We see: this matrix indeed generates rotations!

5.3.3 3D rotations

It is now straightforward to guess the three generators of three-dimensional rotations, which are encoded by the group SO(3). They are

$$\hat{t}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} , \quad \hat{t}^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \quad \hat{t}^3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{15}$$

In the above, $\hat{t}^I$ generates rotations around the x^I-axis. It is easy but a bit tedious to check that indeed

$$\left[\hat{t}^I, \hat{t}^J\right] = \sum_{K=1}^{3} 2\epsilon^{IJK}\hat{t}^K , \tag{16}$$

where $\epsilon^{123} = \epsilon^{231} = \epsilon^{312} = +1$, $\epsilon^{213} = \epsilon^{132} = \epsilon^{321} = -1$, and all other combinations are zero. In particular this means that $f^{IJK} = 2\epsilon^{IJK}$, and hence rotations do not commute and form a non-Abelian group.

5.4 Lie algebra

Lastly, let us mention one more fact about the generators $\hat{t}^I$. As can be seen from their defining equation (7), infinitesimally one can write a group element as

$$g(\alpha) \approx 1 + \sum_{I=1}^{N} \alpha^I \hat{t}^I + \mathcal{O}\left(\alpha^2\right) . \tag{17}$$

This means that locally, around the identity element (which you obtain from setting $\alpha \equiv 0$), the generators form a vector space. This vector space is called the *Lie algebra*. Usually, the Lie algebra of a group $\mathcal{G}$ is denoted as $\mathfrak{g}$ (read: fraktur g) such that $\mathfrak{g} = \mathrm{span}\left\{\hat{t}^I\right\}$. The matrix representations of a Lie algebra $\mathfrak{g}$ and that of the group $\mathcal{G}$ necessarily have the same dimension.

Usually the group definition lets us infer quite a bit about the generators themselves. Take the group O(3) as an example. We know that $g^{-1} = g^{\mathrm{T}}$ (the transposed of a group element is its inverse, as demanded by the orthogonal group). Let us insert this relation, somewhat nonchalantly, into Eq. (7). We obtain:

$$g^{-1} = \left[\exp\left(\sum_{I=1}^{N} \alpha^I \hat{t}^I\right)\right]^{-1} = \exp\left(\sum_{I=1}^{N} -\alpha^I \hat{t}^I\right) \;=\; g^{\mathrm{T}} = \left[\exp\left(\sum_{I=1}^{N} \alpha^I \hat{t}^I\right)\right]^{\mathrm{T}} = \exp\left[\sum_{I=1}^{N} \alpha^I \left(\hat{t}^I\right)^{\mathrm{T}}\right]. \tag{18}$$

Here, $\left(\hat{t}^I\right)^{\mathrm{T}}$ denotes the transposed of the generator (remember that the generator is a matrix). This implies that $\left(\hat{t}^I\right)^{\mathrm{T}} = -\hat{t}^I$, in other words, that the generator has to be an antisymmetric matrix.

Similarly one can show that for the *special* orthogonal group SO(N) the generators have to be antisymmetric matrices that are also tracefree, which stems from the relation

$$\det\left(e^A\right) = e^{\mathrm{Tr}A}. \tag{19}$$

For the unitary group U(N) one finds instead that the generators need to satisfy $\left(\hat{t}^I\right)^{\dagger} = -\hat{t}^I$, that is, the generators need to be anti-Hermitian.[3] Lastly, the generators of the special unitary group SU(N) need to be anti-Hermitian and traceless.

jb, meeting-5-v1.tex, Oct 15, 2018.

References

[1] See the questions 719487 and 1465315 on StackExchange Mathematics (last retrieved Oct 10, 2018). PDFs of these questions are available on our website.

[3]Sometimes one finds the statement that the generators need to be Hermitian. This is not a contradiction, it simply depends on the convention that is used. If we define the group elements as $g(\alpha) = \exp(i \sum \alpha^I \hat{t}^I)$ with an additional factor of "i" in the exponent, then some signs change along the way. It is important to keep track of the convention.