Ising Gauge Theory

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Abstract

This paper investigates a $\mathbb{Z}_2$ lattice gauge theory constructed through a duality mapping from the transverse field Ising model. It is shown that although this system cannot form a local order parameter, it possesses two phases which are confining and deconfining in the $\mathbb{Z}_2$ electric flux respectively. Over the course of this development, we meet much of the machinery used to study more complicated lattice gauge theories.

1. Introduction

Lattice gauge theories are a class of models that arise with increasing frequency in the condensed matter literature in connection to many topics through duality constructions and slave-particle representations. Despite the increasing relevance of lattice gauge theories, pedagogical material on the topic is hard to come by and is often either quite terse or dated. This paper represents my attempt to orient myself in the basics of the lattice gauge theory literature and familiarize myself with some foundational results. The main resources used are the 2013 book by Fradkin [1] whose structure and approach I follow closely throughout this paper, the 1979 review by Kogut [2] and the 2017 book by Shankar [3] whose greater detail and alternate formulation were vital to fill in the gaps left by Fradkin. Of less central use but still vital in my learning process were the references [4, 5, 6].

2. Ising Gauge Theory and Duality

Consider spinors on the vertices of a two-dimensional lattice described by the transverse field Ising model (TFIM). The TFIM Hamiltonian may be written

$$H_{TFIM} = -g \sum_{\langle a\, b\rangle} \sigma^z_a \sigma^z_b - \frac{1}{g}\sum_a \sigma^x_a \quad , \tag{1}$$

where a and b are lattice indices and the constant g describes the strength coupling of the spin degrees of freedom between each-other relative to their coupling to an external magnetic field in the transverse, $\hat{x}$ direction. It is interesting to notice that in two-dimensions, domain walls in the Ising model are finite objects, enclosed by a boundary. If we imagine a 'dual' square lattice which simply interpenetrates the direct spin-lattice, then a domain wall in the direct lattice is a cluster of spins, and its boundary can be thought of as residing in the dual lattice. We may think of the σ^x_a operator as the creation operator for a domain wall which surrounds site a, and $\sigma^z_a\sigma^z_b$ as an operator giving -1 if the link $\langle ab\rangle$ crosses a domain wall and $+1$ otherwise. With this in mind, we introduce operators τ^x_{ij} and τ^z_{ij} on the links of the dual lattice which satisfy the Pauli algebra. We say that the presence of a domain wall on a link corresponds to $\tau^z = -1$ and no domain wall as corresponding to $\tau^z = +1$. Hence we write

$$\sigma^z_a \sigma^z_b = \tau^x_{ij}$$

where $\langle ij\rangle$ intersects $\langle ab\rangle$ and

$$\sigma^x_a = \prod_{\langle ij\rangle \in \square_a} \tau^z_{ij}$$

where $\square_a$ is the plaquette in the dual lattice surrounding the direct lattice site a. We may now write the Hamiltonian in terms of our τ operators as

$$H_{IGT} = -g \sum_{\langle i\,j \rangle} \tau^x_{i\,j} - \frac{1}{g} \sum_a \prod_{\langle i\,j \rangle \,\in \square_a} \tau^z_{i\,j} \tag{2}$$

which is known as the Hamiltonian for *Ising gauge theory*. Because this model can be obtained through a redefinition of the degrees of freedom in the transverse field Ising model, we say that it is *dual* to the TFIM and that they are related via a *duality mapping*.

2.1. Gauge?

One might be puzzled as to why we use the word *gauge* in reference to Eq. (2). First, we recall that the Ising model possesses a *global* $\mathbb{Z}_2$ symmetry where the Hamiltonian is invariant under

$$\sigma^z_a \to -\sigma^z_a$$

for all a in the direct lattice. In contrast with, the Ising gauge theory has a *local* $\mathbb{Z}_2$ symmetry,

$$\tau^z_{ij} \to \eta_i \, \tau^z_{ij} \, \eta_j \tag{3}$$

where $\eta_i = \pm_i 1$ can be chosen arbitrarily at *each* site in the dual lattice. This local symmetry is highly analogous to the local $U(1)$ symmetry of the most well-known gauge theory in physics: electromagnetism. Choosing the value of η_i for each vertex i amounts to choosing a gauge. This gauge freedom arises in the Ising gauge theory because we mapped degrees of freedom on the vertices of a square lattice to the links of another square lattice, but there are more links than vertices in a two dimensional square lattice so we necessarily must introduce redundant degrees of freedom in moving to a theory on the links of the dual lattice.

2.2. The Gauge Constraint

To find the operator generating the symmetry (3), consider that because the Pauli matrices square to the identity, we may write for any site a on the direct lattice

$$\begin{aligned} 1 &= \left(\sigma^z_a \sigma^z_{a+\hat{x}}\right) \left(\sigma^z_{a+\hat{x}} \sigma^z_{a+\hat{x}+\hat{y}}\right) \left(\sigma^z_{a+\hat{x}+\hat{y}} \sigma^z_{a+\hat{y}}\right) \left(\sigma^z_{a+\hat{y}} \sigma^z_a\right) \\ &= \prod_{\langle a\,b \rangle \,\in \square_i} \sigma^z_a \sigma^z_b \\ &= \prod_{j \,\in +_i} \tau^x_{ij} \ , \end{aligned}$$

where $+_i$ is often referred to as the 'star' of site i, ie. the set of links emenating from that site. Defining the last line as G_i, we arrive at our local gauge constant $1 = G_i$ for all sites i. That G_i is indeed a symmetry operator for the $\mathbb{Z}$ gauge theory can be confirmed by showing that $[G_i, H] = 0$ for all i.

Interpreting τ^x as the exponential of another operator

$$\tau^x_{ij} = e^{i\pi E_{ij}} \ ,$$

we see that E_{ij} has eigenvalues 0, 1 mod 2. Hence, the gauge constraint may be written as

$$\begin{aligned} 1 &= G_i \\ &= \tau^x_{i,\,i+\hat{x}} \, \tau^x_{i,\,i-\hat{x}} \, \tau^x_{i\,i+\hat{y}} \, \tau^x_{i,\,i-\hat{y}} \\ &= \tau^x_{i,\,i+\hat{x}} \, \left(\tau^x_{i-\hat{x},\,i}\right)^\dagger \, \tau^x_{i\,i+\hat{y}} \, \left(\tau^x_{i,\,i-\hat{y}}\right)^\dagger \\ &= e^{i\pi(E_{i,\,i+\hat{x}} - E_{i-\hat{x},\,i} + E_{i,\,i+\hat{y}} - E_{i-\hat{y},\,i}} \\ &\equiv e^{i\pi \ \mathbf{\Delta}\cdot\mathbf{E}_i} \end{aligned} \tag{4}$$

where $\mathbf{\Delta} \cdot \mathbf{E}_i$ can be seen as the lattice version of the divergence. Hence, the gauge constraint reduces to Gauß' law in the absense of charge

$$\mathbf{\Delta} \cdot \mathbf{E}_i = 0 \ , \tag{5}$$

for all sites i.

2.3. $\mathbb{Z}_2$ Charge

The general form of Gauß' known from Maxwell electrodynamics is $\nabla \cdot \mathbf{E} = \rho$ where ρ is the charge density. If we imagine a modified version of the TFIM where the nearest neighbor coupling is modulated by some field $B_{ab} = \pm 1$

$$H'_{TFIM} = -g \sum_{\langle a\,b \rangle} B_{ab} \sigma^z_a \sigma^z_b - \frac{1}{g} \sum_a \sigma^x_a \ , \tag{6}$$

then define

$$B_{ab} \sigma^z_a \sigma^z_b = \tau^x_{ij}$$

where link $\langle ij \rangle$ on the dual lattice intersects link $\langle ab \rangle$ on the direct lattice. In this case with the modified definition of τ^x_{ij}, the gauge constraint becomes

$$1 = (-1)^{n_i} \prod_{j \,\in +_i} \tau^x_{ij} = (-1)^{n_i} G_i \tag{7}$$

where n_i is defined as $n_i = 1$ if the dual lattice point i is surrounded by an odd number of direct lattice links $\langle ab \rangle$ with $B_{ab} = -1$, and $n_i = 0$ otherwise. With these definitions the gauge constraint in terms of electric variables becomes

$$\mathbf{\Delta} \cdot \mathbf{E}_i = n_i \quad , \tag{8}$$

suggesting that n_i be interpreted as a quantum of $\mathbb{Z}_2$ charge. Hence, in the physical sector of the Hilbert space

$$\{|\psi\rangle \; s.t. \; (-1)^{n_i} G_i |\psi\rangle = |\psi\rangle \; \forall \; i\} \quad ,$$

any site i with $n_i = 1 \bmod 2$ must have an odd number of $\tau^x = -1$ links emenating from it.

3. Phases of the Ising Gauge Theory

Elitzur's theorem, states that there can be no spontaneously broken symmetries in theories with local gauge invariance[7]. One way of seeing this (shown in Appendix A) is to take seriously the gauge degrees of freedom as physical objects and show that freedom under gauge transformation forbids the formation of a local order parameter. Another way of convincing oneself of Elitzur's result is to realize that local gauge symmetry is just a redundancy in our description of a system arising from mapping a system with some number of degrees of freedom to another system with a larger number of degrees of freedom. In such a light, it can be said to be *obvious* that one cannot spontaneously break a local symmetry as it was never physical to begin with.

Whichever philosophy one takes, Elitzur's theorem shows us that if there is a phase transition, as indeed we know there is from studies of the conventional TFIM, then they must form without a local order parameter.

3.1. The Strong Coupling Limit

Consider the large g limit in which

$$H \approx -g \sum_{\langle ij \rangle} \tau^x_{ij}$$

meaning that in the limit as $g \to \infty$ the ground state is an eigenstate of τ^x,

$$\tau^x_{ij} \, |\Phi\rangle_{g\to\infty} = + \, |\Phi\rangle_{g\to\infty} \quad \forall \, \langle ij \rangle \quad .$$

3.1.1 Confinement

In the strong coupling limit where $\langle \tau^x_{ij} \rangle = +1$, we see that this implies $\langle E_{ij} \rangle = 0$ meaning that there are no electric fields. If we were to insert by hand two $\mathbb{Z}_2$ charges at points k and l separated by L links on the dual lattice, we know that from gauge invariance there must be an odd number of $\tau^x = -1$ links emanating from each such point and an even number everywhere else. Hence, there must be a string (electric flux line) of $\tau^x = -1$ links with endpoints k and l and since each flipped link costs $2g$ quanta of energy, the system must pay an energy penalty of $2gL$ in order to keep the charges separated a distance L.

Because of this large energy pentalty for separated charges, we call the strong coupling limit of the Ising gauge theory a $\mathbb{Z}_2$ *confined* phase.

3.1.2 The Wilson Wegner Loop - Strong Coupling

Because $\mathbb{Z}_2$ lattice gauge theories do not have local order paramters, we will find it useful to construct a non-local, gauge invariant operator known as the Wilson-Wegner loop operator

$$W_\Gamma = \prod_{\langle ij \rangle \, \in \, \Gamma} \tau^z_{ij} \quad , \tag{9}$$

where Γ is a closed loop in the dual lattice. W_Γ can be seen as a generalization of the plaquette operator from the Hamiltonian (2) and it can be thought of as creating a Ising domain wall with boundary Γ. We see that for any Γ in the infinite coupling limit, its expectation value vanishes

$$\langle \Phi | W_\Gamma | \Phi \rangle_{g\to\infty} = 0$$

because τ^z acting on a τ^x eigenstate will flip that eigenstate to the opposite τ^x eigenstate. Taking $g \gg 1$ but finite, we can expand $|\Phi\rangle$ in powers of the plaquette term as shown in Appendix B to find

$$\langle \Phi | W_\Gamma | \Phi \rangle_{g \gg 1} \propto e^{-2 \ln g \; A[\Gamma]}$$

where $A[\Gamma]$ is the number of plaquettes enclosed by the curve Γ, ie. its *area*.

3.2. The Weak Coupling Limit

We now turn to the weak coupling limit where $g \ll 1$ in which the Hamiltonian may be approximated as

$$H_{IGT} \approx -\frac{1}{g}\sum_a \prod_{\langle ij\rangle \in \square_a} \tau^z_{ij} \quad .$$

The ground state in this weak coupling limit must be an eigenstate of the plaquette operator

$$P_a = \prod_{\langle ij\rangle \in \square_a} \tau^z_{ij} \quad ,$$

at all direct lattice sites a. Constructing a vacuum state $|0\rangle$, defined as the state where all links have $\tau^x = +1$, one can write the $+1$ eigenstate of P_a as

$$|P_a = +1\rangle = (1 + P_a)\,|0\rangle \quad ,$$

because $P_a^2 = \mathbb{1}$ due to the Pauli algebra. Likewise, the -1 eigenstate would be written

$$|P_a = -1\rangle = (1 - P_a)\,|0\rangle \quad .$$

Comparing to the Hamiltonian, having a single plaquette in the $P_a = -1$ eigenstate would introduce an energy cost of $2/g$, meaning the spctrum is *gapped*, just as in the strong coupling case. For any state $|\psi\rangle$, a state with lower energy can be created by adding it in linear superpositon with the state $W_\Gamma\,|\psi\rangle$ for any W_Γ.

A state which minimizes the $g \to 0$ Hamiltonian, is

$$|\Phi\rangle_{g\to 0} = 2^{-N}\prod_a (1 + P_a)\,|0\rangle \tag{10}$$

where N is the number of plaquettes. This state $|\Phi\rangle_{g\to 0}$, is often called a *loop gas* or *string net condensate* as it is an equal phase superposition of closed electric flux lines of arbitrary length. That this state minimizes the Hamiltonian may be seen inductively, starting with a variational ground state $|\Psi\rangle$ equal to the $\tau^x = +1$ vacuum which has zero energy expectation value and then showing that $|\tilde{\Psi}\rangle = |\Psi\rangle + W_\Gamma\,|\Psi\rangle$ has a lower energy for any closed contour Γ which is not already in the superposition. One can then proceed iteratively to find that the optimal ground state is the equal weight superposition of all closed contours in Eq. (10).

3.2.1 Deconfinement

If we again imagine placing two $\mathbb{Z}_2$ charges on the lattice separated by some distance, the vacuum state $|\tilde{0}\rangle$, is now simply all spins in the $\tau^x = +1$ eigenstate except along a line joining the two charges where the spins must be in the -1 eigenstate to satisfy gauge invariance. The ground state is again constructed in the same way as before

$$|\tilde{\Phi}\rangle = \prod_a (1 + P_a)\,|\tilde{0}\rangle \quad ,$$

and now the plaquette operators not only create closed loops on the vacuum but change the shape of the flux line connecting the two charges arbitrarily as shown in Figure 1. Hence, in the limit $g \to 0$, there is no cost for having two charges separated at a finite distance and electric flux lines proliferate through the lattice, motivating us to call the weak coupling phase *deconfining*.

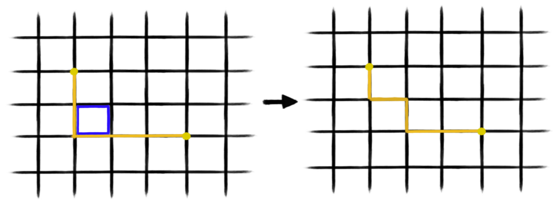

Figure 1: *Two $\mathbb{Z}_2$ charges connected by a flux line (yellow) being deformed by an insertion of a plaquette operator (blue).*

3.2.2 The Wilson Wegner Loop - Weak Coupling

In the limit of small but finite g, if one perturbatively expands the ground state now in powers of g^2 as shown in Appendix C, the Wison-Wegner loop operator will have expectation value

$$\langle G|W_\Gamma|G\rangle_{g\ll 1} = e^{-g^4 L[\Gamma]/8}$$

where $L[\Gamma]$ is the perimiter of Γ.

This shift from the expectation value of the Wilson-Wegner loop following an area law to a perimiter law, as we go from strong to weak coupling is evidence for a *phase transition* between the confined and deconfined phases as crossing over from one behaviour to another typically requires non-analyticity[8].

4. Conclusion

While this paper with its focus on the Ising gauge theory has only scratched the surface of lattice gauge theories in general, it presents a simple minimal example of how to construct a lattice gauge theory from a duality mapping, how to represent charges in that gauge theory and how to probe its phase diagram while avoiding the problems posed by gauge symmetry. While many of the details will not apply to more rich lattice gauge theories, the general thrust of the techniques employed *are* applicable and it is my hope that this paper can serve as a rough map as I explore further afield into more exotic lattice gauge theories.

References

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Appendices

A. Elitzur's Theorem

As shown by Elitzur in 1975 [7], a field subject to a local symmetry cannot undergo spontaneous symmetry breaking which in the case of Ising gauge theory, means that τ^z cannot magnetize. Following Kogut's treatment [2], we add to the Hamiltonian (2) a term biasing τ^z with an external field

$$H_h = H_{IGT} + h \sum_{\langle i,j \rangle} \tau^z_{ij} \quad .$$

The magnetiziation of the gauge field τ^z on link ij in response to the external field is given by

$$\langle \tau^z_{ij} \rangle_h = \frac{1}{Z} \sum_{\{\tau^z\}} \tau^z_{ij} \exp \left[-\beta \left(H_{IGT} + h \sum_{\langle kl \rangle} \tau^z_{kl} \right) \right] \quad . \tag{11}$$

We now perform a local gauge transfromation at site n, ie. $\eta_k = -1$ for $k = i$, and 1 otherwise. We have already shown that this transformation is a symmetry of H_{IGT}, but the biasing field will transform as

$$h \sum_{\langle kl \rangle} \tau^z_{kl} \rightarrow h \sum_{\langle ij \rangle} \tau^z_{ij} - 2h \sum_{\langle ij \rangle \in +_i} \tau^z_{ij}$$

and hence the magnetization becomes

$$\begin{aligned} \langle \tau^z_{ij} \rangle_h \rightarrow & \frac{1}{Z} \sum_{\{\tau^z\}} -\tau^z_{ij} \exp \left[-\beta \left(H_h - 2h \sum_{\langle ij \rangle \in +_i} \tau^z_{ij} \right) \right] \\ = & \left\langle -\tau^z_{ij} \, e^{2\beta h \sum_{\langle ij \rangle \in +_i} \tau^z_{ij}} \right\rangle_h \quad . \end{aligned}$$

Recall that the *spontaneous* magnetization of a field is found by taking the thermodynamic limit of the magnetization (11), followed by the limit $h \rightarrow 0$. Now we may use the fact that the exponential in the second line depends only on the links emenating from site i and is hence unchanged in the thermodynamic limit. This means that we may safely send h to zero, implying that

$$\langle \tau^z_{ij} \rangle_{h \rightarrow 0} = 0 \quad ,$$

as required by Elitzur's theorem. This example showcases why local gauge invariance is essential to this argument because if the gauge transformation caused an *extensive* shift in the biasing field (corresponding to a global gauge transformation) the argument would break down and a spontaneous magnetization would be allowed. This result holds even at zero temperature [1].

B. Strong Coupling Expansion of the Wilson-Wegner Loop Operator

In the strong coupling limit, we can perturbatively expand the ground state $|\Phi\rangle$ around the point $g = \infty$ as

$$|\Phi\rangle = |\Phi_0\rangle + \left(\frac{H_P}{\Delta E} \right) |\Phi_0\rangle + \left(\frac{H_P}{\Delta E} \right)^2 |\Phi_0\rangle + ...$$

where the energy cost of each plaquette ΔE is of order g (8g for an isolated plaquette), $|\Phi_0\rangle = \prod_{\langle ij \rangle} |\tau^x_{ij} = 1\rangle$ and H_P is the plaquette term of the Hamiltonian

$$H_p = \frac{1}{g} \sum_{\square} \prod_{\langle ij \rangle \in \square} \tau^z_{ij} \quad ,$$

and thus this is an expansion in powers of $1/g^2$.

Expanding the expectation value of the Wilson loop operator (9) in the perturbation series, we see

$$\begin{aligned} \langle \Phi_0 | W_\Gamma | \Phi_0 \rangle_{g \gg 1} = & \langle \Phi_0 | W_\Gamma | \Phi_0 \rangle + \langle \Phi_0 | W_\Gamma \left(\frac{H_P}{\Delta E} \right) |\Phi_0\rangle \\ & + \langle \Phi_0 | W_\Gamma \left(\frac{H_P}{\Delta E} \right)^2 |\Phi_0\rangle + \ ... \end{aligned}$$

In order to get a non-zero overlap in any term in the above expansion, every flipped link in the Wilson loop operator must be 'un-flipped' by H_P. Each power of H_P can flip the spins around *one* plaquette, but the product of neighboring plaquette operators will make a loop of flipped spins which encloses both of them as the spins on the interface *between* the two plaquettes will be flipped twice, giving the identity as shown in Figure 2. Hence, if Γ is a closed loop enveloping n plaquettes, the above perturbative expansion will vanish up to order n where there are n factors of H_P to make a loop which undoes the action of W_Γ. Hence,

$$\langle \Phi | W_\Gamma | \Phi \rangle_{g \gg 1} \propto \left(\frac{1}{g^2} \right)^n = e^{-2 \ln g \, A[\Gamma]}$$

where $A[\Gamma] = n$ is the area encolosed by Γ expressed in the number of plaquettes it encloses.

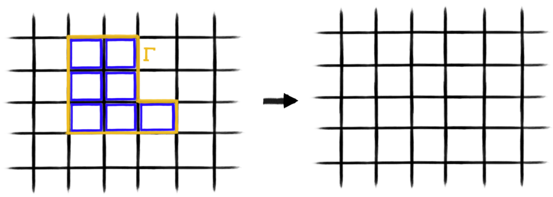

Figure 2: *Action of n plaquette operators (blue squares) cancelling the action of a W_Γ operator (yellow) enclosing the n plaquettes.*

C. Weak Coupling Expansion of the Wilson-Wegner Loop Operator

In working with the Wilson-Wegener loop at weak coupling, we will find it useful to instead work in the τ^z basis. The naive trial ground state in such a representation would be the τ^z vacuum, $|0_z\rangle$ where $\tau^z = +1$ everywhere but for gauge invariance, we must take the vacuum in a sum with all states related to it by a gauge transformation

$$|\Phi\rangle_{g\to 0} = \frac{1}{\sqrt{\mathcal{V}}}\sum_\alpha g_\alpha |0\rangle \quad ,$$

where $\mathcal{V}$ is the multiplicity of all possibly gauge transformations and g_α are cominbations of G_i. In this basis, we can write the expectation value of the domain wall operator as

$$\langle\Phi|W_\Gamma|\Phi\rangle_{g\to 0} = \frac{1}{\mathcal{V}}\sum_{\alpha\beta}\langle 0|g_\alpha W_\Gamma g_\beta|0\rangle$$

and now we may interpret

$$\frac{1}{\mathcal{V}}\sum_{\alpha\beta} g_\alpha W_\Gamma g_\beta$$

as a gauge transfromation on W_Γ, but W_Γ is a *gauge invariant* and so we have

$$\begin{aligned}\langle\Phi|W_\Gamma|\Phi\rangle_{g\to 0} &= \langle 0|W_\Gamma|0\rangle \\ &= 1 \quad .\end{aligned}$$

Now if we expand the ground state perturbatively including factors of $g\sum_{\langle ij\rangle}\tau^z_{ij}$ which introduce an energy cost of $4g$ by frustrating the two plaquettes sharing bond $\langle ij\rangle$, we see

$$|\Phi\rangle_{g\ll 1} = |\Phi_0\rangle + \frac{g^2}{4}\sum_{\langle ij\rangle}\sigma_{ij}|\Phi_0\rangle \quad .$$

If we let W_Γ act on this state, the presense of the τ^x_{ij} will flip the sign *if* bond $\langle ij\rangle$ lies on Γ, so we may write

$$\begin{aligned}W_\Gamma|\Phi\rangle_{g\ll 1} = |\Phi_0\rangle &+ \frac{g^2}{4}\sum_{\langle ij\rangle\notin\Gamma}\tau^x_{ij}|\Phi_0\rangle \\ &- \frac{g^2}{4}\sum_{\langle ij\rangle\in\Gamma}\tau^x_{ij}|\Phi_0\rangle \quad ,\end{aligned}$$

such that

$$\begin{aligned}\frac{\langle\Phi|W_\Gamma|\Phi\rangle}{\langle\Phi|\Phi\rangle} &= \frac{1+\frac{g^4}{16}(N-2L)}{1+g^4N/16} \\ &= 1 - \frac{g^4L}{8} + \ldots \\ &\approx e^{-g^4L/8}\end{aligned}$$

where L is the number of links in Γ and N is the number of links on the lattice. This development closely follows that in Shankar, 2017 [3].